Lecture 25 Laplace transform, the real Fourier transform

Laplace transform: $f(x)$ on $[0,\infty) \longmapsto \underbrace{F(s)}_{\text{function of } s \text{ on } s>a.} = \int_0^\infty f(t)e^{-st}\,dt$

Useful for solving ODEs (initial value problems at $t=0$)

Big facts 1) $\mathcal{L}[f(t)] = F(s)$ is a linear operator.

2) $\mathcal{L}[f'] = sF(s) - f(0)$

3) $\mathcal{L}[f''] = s\mathcal{L}[f'] - f'(0) = s^2F(s) - f(0)s - f'(0)$

Why 2: $\mathcal{L}[f'] = \int_0^\infty f'(t)\,e^{-st}\,dt$

$$= \lim_{B\to\infty} \int_0^B \underbrace{e^{-st}}_{u}\,\underbrace{f'(t)\,dt}_{dv}$$

$du = -se^{-st}dt$

$v = f(t)$

$$= \lim_{B\to\infty}\left(e^{-st}f(t)\Big|_0^B - \underbrace{\int_0^B f(t)\left[-se^{-st}\right]dt}_{s\int_0^B f(t)e^{-st}dt} \right)$$

need $f(t)$ to be of "exponential type", meaning that $|f(t)| \le Me^{kt}$ for some $M, k > 0$.

$$= \underset{\text{when } s>k}{0} - f(0) + sF(s) \checkmark$$

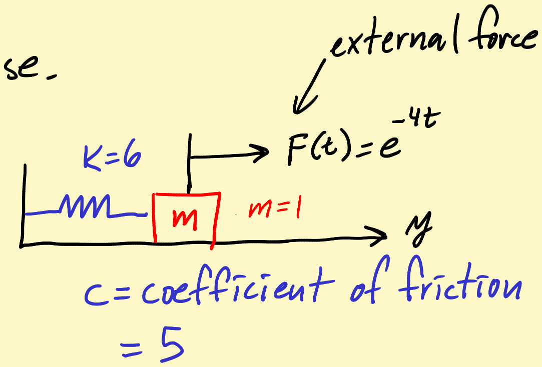
Also need f C^1-smooth on $[0,\infty)$.

4) Can "undo" the Laplace transform!

If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > M$ (for some M),

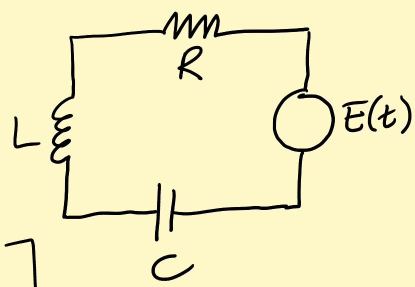
then $f_1 \equiv f_2$ on $[0, \infty)$.

Consequently, $\mathcal{L}^{-1}$ makes sense.



<u>EX</u> $\quad y'' + 5y' + 6y = e^{-4t}$

$$\begin{cases} y(0) = 2 \\ y'(0) = 3 \end{cases}$$



Hit ODE with $\mathcal{L}$

$$\left[s^2 Y(s) - s\overbrace{y(0)}^{2} - \overbrace{y'(0)}^{3}\right] + \left[5\left(sY(s) - \overbrace{y(0)}^{2}\right)\right] + 6Y(s) = \frac{1}{s+4}$$

$$Y(s)\left(s^2 + 5s + 6\right) = 2s + 13 + \frac{1}{s+4}$$

$$Y(s) = \underbrace{\frac{2s+13}{s^2+5s+6}}_{\text{sol}^n \text{ to homog}} + \underbrace{\frac{1}{(s+4)(s^2+5s+6)}}_{\substack{\text{particular} \\ \text{sol}^n \text{ to ODE}}}$$

Apply $\mathcal{L}^{-1}$: $\quad y(t) = c_1 e^{-2t} + c_2 e^{-3t}$

How it feels: Case $\begin{cases} y(0) = 0 \\ y'(0) = 0. \end{cases}$

$$Y(s) = \frac{1}{(s+4)(s+3)(s+2)}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$y(t) = \underbrace{Ae^{-2t} + Be^{-3t}}_{\text{homog sol}^n} + \underbrace{Ce^{-4t}}_{\text{particular sol}^n}$$

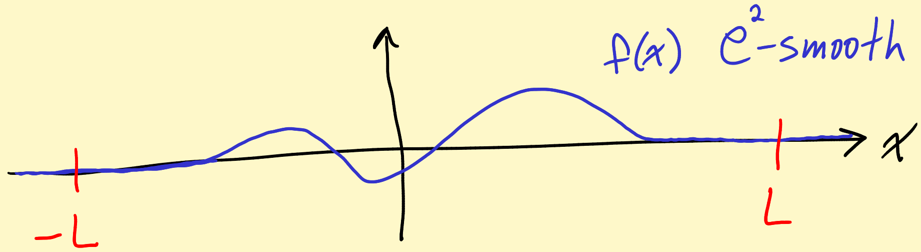
EXCITING THING Works great for piecewise defined fcns!

Great tool $\mathcal{L}$: ODE's $\longrightarrow$ Algebra

Solve algebra for $Y(s)$.

Get sol^n $y(t) = \mathcal{L}^{-1}[Y(s)]$.

The real Fourier integral. Motivation:



$f(x)$ C^2-smooth with "compact support"

$\operatorname{Supp} f$ = Closure of the set $\{x : f(x) \neq 0\}$.

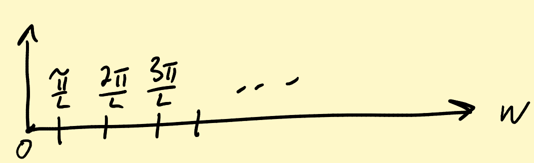
$\operatorname{Supp} f$ compact means this is closed and bounded.
always is closed

Idea:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$= \underbrace{\left(\frac{1}{2L}\int_{-L}^{L} f(t)\,dt\right)}_{\to 0 \text{ as } L\to\infty} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\overbrace{\left(\underbrace{\frac{1\cdot\pi}{L}}_{\Delta w}\int_{-L}^{L} f(t)\cos\underbrace{\frac{n\pi}{L}}_{w_n} t\,dt\right)}^{a_n}\cos\frac{n\pi x}{L} + \text{Sine stuff}\right]$$

= Riemann sum in w's



$$= \sum_{n=1}^{\infty} \frac{1}{\pi} \left(\int_{-\infty}^{\infty} f(t) \cos w_n t \, dt \cdot \cos w_n x \cdot \Delta w + \begin{matrix} \text{Sine} \\ \text{stuff} \end{matrix} \right)$$

Can replace L by ∞ since f has compact support

$$\xrightarrow[L \to \infty]{} \int_0^{\infty} A(w) \cos wx + B(w) \sin wx \, dw$$

$$\text{where} \begin{cases} A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos wt \, dt \\ B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin wt \, dt \end{cases}$$

Can be made rigorous!

Summary:

$$f(x) = \overbrace{\int_0^{\infty} A(w) \cos wx + B(w) \sin wx \, dw}^{\text{Fourier integral}}$$

$$\text{where} \begin{cases} A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos wt \, dt \\ B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin wt \, dt \end{cases}$$

Feeling $f(x)$ = limit of Riemann sums

= limit of linear combo's of $\cos wx$, $\sin wx$

as w ranges over <u>all</u> positive frequencies!

Fourier series: $f(x)$ 2π-periodic can be expanded as a linear combo of "musical frequencies" $\cos nx$, $\sin nx$