Lecture 26 Even & odd fcns, the Fourier cosine and sine transforms

HWK 6 due Thurs

Fourier integral:

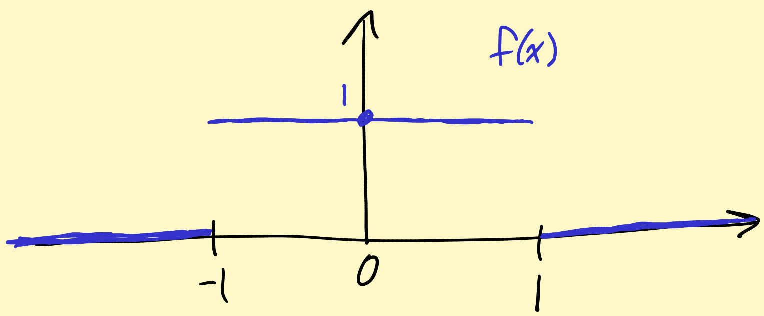
$$f(x) = \int_0^\infty A(w)\cos wx + B(w)\sin wx \, dw$$

where
$$\begin{cases} A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt \, dt \\ B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt \, dt \end{cases}$$

__Fact__ If f is piecewise C^1-smooth and $\int_{-\infty}^{\infty} |f| \, dx$ is finite.

Then $f(x)$ "=" its Fourier integral $= \begin{cases} f(x) & \text{where } f \text{ continuous} \\ \text{midpoint of jumps} & \text{at jumps} \end{cases}$

__EX__



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{even}} \underbrace{\cos wt}_{\text{even}}}_{\text{even}} \, dt = \frac{2}{\pi}\int_0^\infty f(t)\cos wt \, dt$$

$$= \frac{2}{\pi}\int_0^1 1 \cdot \cos wt \, dt + 0$$

$$= \frac{2}{\pi}\left[\frac{1}{w}\sin wt\right]_0^1$$

$$A(w) = \frac{2}{\pi} \frac{\sin w}{w}$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{even}}\ \underbrace{\sin wt}_{\text{odd}}}_{\text{odd}}\, dt = 0$$

Note: $\int_{-\infty}^{\infty} = \lim_{A\to\infty} \int_{-A}^{A}$

because f is absolutely integrable

$$f(x) = \boxed{\int_0^{\infty} \frac{2}{\pi} \frac{\sin w}{w} \cos wx \, dw}$$

$$= \begin{cases} 0 & x < -1 \\ \frac{1}{2} & x = -1 \\ 1 & -1 < x < 1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$$

Hmmm. Let $x = 0$. $\quad 1 = f(0) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin w}{w} \cdot 1 \, dw$

Aha! $\quad \int_0^{\infty} \frac{\sin w}{w} \, dw = \frac{\pi}{2}$

even

$$\int_{-\infty}^{\infty} \frac{\sin w}{w} \, dw = \pi$$

Fourier cosine & sine transforms

Step 1 What happens to Fourier integral when $f(x)$ is even or odd.

f(x) even

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\overset{\text{even}}{f(t)}\ \overset{\text{even}}{\cos wt}}_{\text{even}}\, dt = \frac{2}{\pi}\int_0^{\infty} f(t)\cos wt\, dt$$

$$B(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{(\text{even})(\text{odd})}_{\text{odd}}\, dt = 0$$

Hmmm.

$$f(x) = \int_0^{\infty} A(w)\cos wx\, dx$$

where $A(w) = \frac{2}{\pi}\int_0^{\infty} f(t)\cos wt\, dt$

Fourier cosine transform

$$\tilde{\mathcal{F}}_c[f] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\, dx$$

$$\tilde{\mathcal{F}}_c : (\text{function on } [0,\infty)) \longmapsto (\text{function on } [0,\infty))$$

Fantastic thing: $\boxed{\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c = \text{identity} \ !}$

(To understand, we needed to extend f to $(-\infty, \infty)$ as an even fcn and use Fourier integral.)

Repeat ideas for $f(x)$ odd. Get Fourier sine transform

$$\tilde{\mathcal{F}}_s[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

Big fact: $\tilde{\mathcal{F}}_s\left[\tilde{\mathcal{F}}_s[f]\right] = f$

<u>Other key properties</u>

$$\tilde{\mathcal{F}}_c[f'] = w \, \tilde{\mathcal{F}}_s[f] - \sqrt{\frac{2}{\pi}} f(0)$$

need $f(x) \to 0$ as $x \to \infty$.

$$\tilde{\mathcal{F}}_s[f'] = -w \, \tilde{\mathcal{F}}_c[f]$$

Do these again:

$$\tilde{\mathcal{F}}_c[f''] = -w^2 \, \tilde{\mathcal{F}}_c[f] - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\tilde{\mathcal{F}}_s[f''] = -w^2 \, \tilde{\mathcal{F}}_s[f] + \sqrt{\frac{2}{\pi}} f(0) w$$

<u>Why</u> $\tilde{\mathcal{F}}_c[f'] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx$

$$= \lim_{A \to \infty} \sqrt{\frac{2}{\pi}} \int_0^A \underbrace{\cos wx}_{u} \; \underbrace{f'(x)\, dx}_{dv}$$

$du = -w \sin wx \, dx$

$v = f(x)$

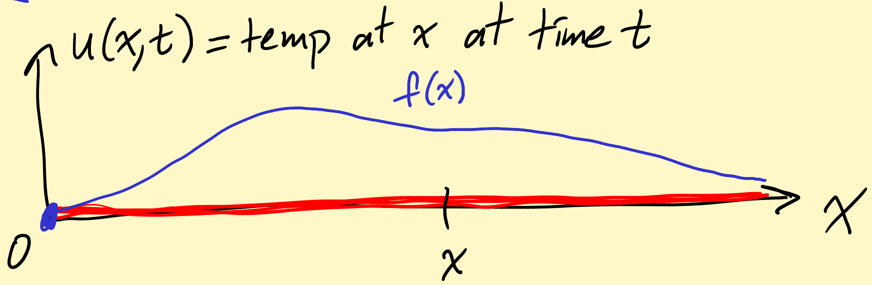
$$= \lim_{A\to\infty} \sqrt{\tfrac{2}{\pi}} \left[\left(\cos wx\right) f(x) \Big|_0^A - \int_0^A f(x) \left[-w \sin wx \, dx\right] \right]$$

$$= \sqrt{\tfrac{2}{\pi}} \left[0 - 1 \cdot f(0) \right] + \sqrt{\tfrac{2}{\pi}}\, w \int_0^\infty f(x) \sin wx \, dx$$

because $f(A) \to 0$ as $A \to \infty$

($\tilde{\mathcal{F}}_s[f']$ is easier because $\sin 0 = 0$.)

Application Heat problem on semi-infinite hot wire



Heat eqn $\qquad \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

Initial condition: $\quad u(x,0) = f(x)$

Boundary condition: $\quad u(0,t) = 0 \leftarrow$ homog bndry cond

Big idea: Apply $\tilde{\mathcal{F}}_s$ in space variable x.

Write $\quad \hat{u}(w,t) = \sqrt{\tfrac{2}{\pi}} \displaystyle\int_0^\infty u(x,t) \sin wx \, dx$

PDE?

$$\frac{\partial \hat{u}}{\partial t}(w,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial u}{\partial t}(x,t)}_{c^2 \frac{\partial^2 u}{\partial x^2}} \sin wx \, dx$$

$\leftarrow$ take $c=1$.

$$= \mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$= -w^2 \mathcal{F}_s[u] + \sqrt{\frac{2}{\pi}} \underbrace{u(0,t)}_{\substack{=0 \\ \text{B.C.}}}$$

Aha!

$$\boxed{\frac{\partial \hat{u}}{\partial t} = -w^2 \hat{u}}$$

$\leftarrow$ turn PDE into an "ODE"!