Lecture 27 Applications of Fourier sine & cosine transforms

$$\tilde{\mathcal{F}}_c[f] = \sqrt{\tfrac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

$$\tilde{\mathcal{F}}_s[f] = \sqrt{\tfrac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

fcns of w on $[0,\infty)$

Important properties: $\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c = id$, $\tilde{\mathcal{F}}_s \circ \tilde{\mathcal{F}}_s = id$

$$\begin{cases} \tilde{\mathcal{F}}_c[f'] = w \tilde{\mathcal{F}}_s[f] - \sqrt{\tfrac{2}{\pi}} f(0) \\ \tilde{\mathcal{F}}_s[f'] = -w \tilde{\mathcal{F}}_c[f] \end{cases}$$

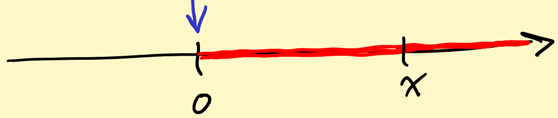
$$\begin{cases} \tilde{\mathcal{F}}_c[f''] = -w^2 \tilde{\mathcal{F}}_c[f] - \sqrt{\tfrac{2}{\pi}} f'(0) \\ \tilde{\mathcal{F}}_s[f''] = -w^2 \tilde{\mathcal{F}}_s[f] + \sqrt{\tfrac{2}{\pi}} f(0) w \end{cases}$$

good for insulated end

good for iced end (homog BC)

Heat problem

temp held at 0



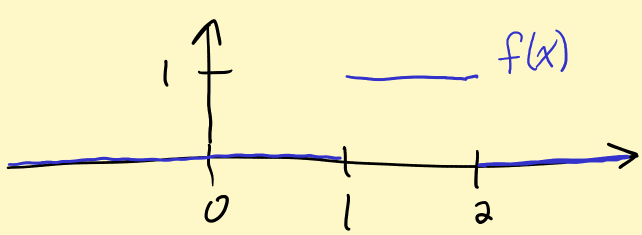
wire $[0,\infty)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{PDE (Heat eqn)}$$

$$u(0,t) = 0 \qquad \text{BC (homog)}$$

$$u(x,0) = f(x) \leftarrow \text{given} \qquad \text{IC}$$

Take



$$f(x) = \begin{cases} 0 & x<1 \\ 1 & 1<x<2 \\ 0 & x>2 \end{cases}$$

Big idea: Apply $\tilde{\mathcal{F}}_s$ in <u>space</u> variable x.

$$\hat{u}(w,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin wx \, dx$$

PDE: $$\frac{\partial \hat{u}(w,t)}{\partial t} = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial u}{\partial t}(x,t)}_{\frac{\partial^2 u}{\partial x^2}} \sin wx \, dx$$

$$= \tilde{\mathcal{F}}_s\left[\frac{\partial^2 u}{\partial x^2}\right](w)$$

$$= -w^2 \underbrace{\tilde{\mathcal{F}}_s[u](w)}_{\hat{u}(w,t)} + \sqrt{\frac{2}{\pi}} \underbrace{u(0,t)}_{=0} w$$

$= 0 \leftarrow$ why we chose $\tilde{\mathcal{F}}_s$ instead of $\tilde{\mathcal{F}}_c$

$$\boxed{\frac{\partial \hat{u}}{\partial t} = -w^2 \hat{u}}$$

$\leftarrow$ easy "ODE" in t

$$\hat{u} = C e^{-w^2 t}$$

hmmm. C might depend on w.

Why $\frac{\partial}{\partial t}\left[\frac{\hat{u}}{e^{-w^2 t}}\right] \equiv 0$. So $\frac{\hat{u}}{e^{-w^2 t}}$ is a fcn of other var w. Call it $C(w)$.

Get $$\boxed{\hat{u}(w,t) = C(w)\, e^{-w^2 t}}$$

How to determine $C(w)$: Use the Initial condition!

$$\hat{u}(w,0) = C(w) \cdot \underbrace{e^0}_{1}$$

$$\hat{u}(w,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{u(x,0)}_{f(x)} \sin wx \; dx = \underbrace{\tilde{\mathcal{F}}_s[f(x)]}_{C(w)}$$

$$\boxed{C(w) = \tilde{\mathcal{F}}_s[f(x)]}$$

$$\hat{u}(w,0) = \sqrt{\frac{2}{\pi}} \int_1^2 1 \cdot \sin wx \; dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{1}{w} \cos wx \right]_{x=1}^{2}$$

$$= \sqrt{\frac{2}{w}} \left(\frac{\cos w - \cos 2w}{w} \right) = C(w)$$

Have $\hat{u}(w,t) = C(w) e^{-w^2 t}$ where $C(w) =$

Finally, "undo" $\tilde{\mathcal{F}}_s$ by applying it again!

$$u(x,t) = \tilde{\mathcal{F}}_s \left[C(w) e^{-w^2 t} \right]$$ ← apply $\tilde{\mathcal{F}}_s$ in w variable

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right) e^{-w^2 t} \sin xw \; dw$$

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \left(\frac{\cos w - \cos 2w}{w} \right) e^{-w^2 t} \sin wx \; dw$$

sol^n!

L'H shows $\lim_{w \to 0} \dfrac{\cos w - \cos 2w}{w} = 0$

$$\frac{\cos w - \cos 2w}{w} = \frac{\left(1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \cdots\right) - \left(1 - \frac{(2w)^2}{2!} + \frac{(2w)^4}{4!} - \cdots\right)}{w}$$

$$= \frac{\left(\frac{3}{2}\right)w^2 + \cdots}{w} = \left(\frac{3}{2}\right)w + \text{higher order terms}$$

Radius of conv of series $= \infty$.

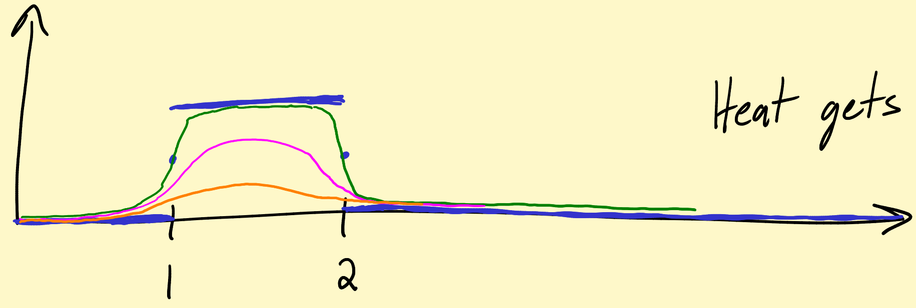
$= 0$ when $w = 0$ because no constant.
Infinitely diff'ble!

$\dfrac{\cos w - \cos 2w}{w} \to 0$ as $w \to \infty$ too.

$e^{-w^2 t} \to 0$ as $w \to \infty$ very fast (when $t > 0$).

$\sin wx$ is bounded. Integral is very well behaved!

Can verify that this really solves problem!



Heat gets sucked away very rapidly!

Case of insulated end : No heat flow past $x=0$ end.

Kelvin's law: $\underbrace{(\text{Rate of heat flow})}_{0} = -c\,(\text{temp gradient})$

$= -c \dfrac{\partial u}{\partial x}(0,t)$

Insulated end BC : $\boxed{\frac{\partial u}{\partial x}(0,t) = 0}$

Use $\tilde{\mathcal{F}}_c$ instead because

$$\tilde{\mathcal{F}}_c[f''] = -w^2\, \tilde{\mathcal{F}}_c[f] - \sqrt{\frac{2}{\pi}}\, \underbrace{f'(0)}_{\substack{BC \\ =0}}$$

<u>Next time</u> Complex Fourier transform

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

Fourier inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx}\, ds$$

Write $\hat{f}(s) = \tilde{\mathcal{F}}[f]$.

$$\tilde{\mathcal{F}}[f'] = is\, \tilde{\mathcal{F}}[f]$$

$$\tilde{\mathcal{F}}[f''] = -s^2\, \tilde{\mathcal{F}}[f]$$