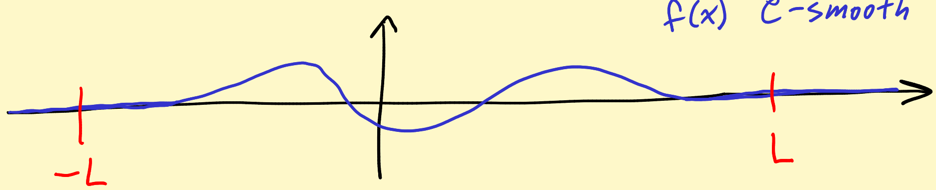
Lecture 28 The complex version of the Fourier transform

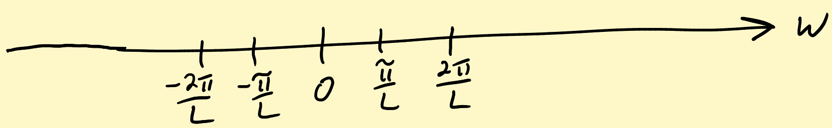


$f(x)$ C^2-smooth with compact support in $(-L,L)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-in\pi t/L}\, dt$$

$$= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Bigg(\frac{1\cdot \pi}{2L} \int_{\underbrace{-L}_{-\infty}}^{\overbrace{L}^{\infty}} f(t)\, e^{-i \underbrace{\frac{n\pi}{L}}_{w_n} t}\, dt \Bigg) e^{i \underbrace{\frac{n\pi}{L}}_{w_n} x}$$

$\frac{\pi}{L} = \Delta w$



$$f(x) = \xrightarrow[L\to\infty]{} \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt \right)}_{\text{Fourier transform}} e^{iwx}\, dw$$

where to put?

Conventions

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

also write $\hat{f}(s) = \tilde{\mathcal{F}}[f]$

Fourier inversion formula $\quad f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx}\, ds$

Also write $\displaystyle \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{isx}\, ds$

$$= \mathcal{F}^{-1}[g]$$

Inversion formula: $f = \check{\hat{f}} = \mathcal{F}^{-1}\left[\mathcal{F}[f]\right]$

Big properties

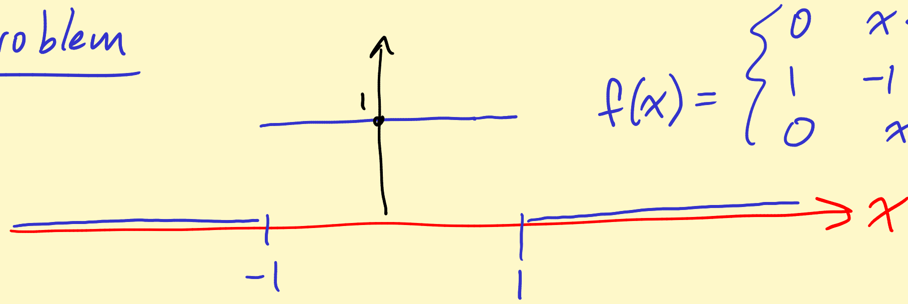
1) $\mathcal{F}[f'] = is\,\hat{f}(s)$

2) $\mathcal{F}[f''] = -s^2\,\hat{f}(s)$

3) If f is piecewise C^1-smooth and $\int_{-\infty}^{\infty} |f|\,dx < \infty$,

then
$$f(x) \text{ ``$=$'' } \mathcal{F}^{-1}[\hat{f}] = \begin{cases} f(x) & f \text{ cont at } x \\ \text{midpoint of jump} & \text{at jumps} \end{cases}$$

Hot wire problem



$$f(x) = \begin{cases} 0 & x < -1 \\ 1 & -1 < x < 1 \\ 0 & x > 1 \end{cases}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (c=1)$$

$$u(x,0) = f(x)$$

(No boundary. So no boundary conditions.)

Big idea: Apply $\mathcal{F}$ in space variable x.

Write $\quad \hat{u}(s,t) = \mathcal{F}[u] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-isx}\, dx$

Hmmm. Take $\frac{\partial}{\partial t}$ of this

$$\frac{\partial \hat{u}(s,t)}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{\partial u}{\partial t}(x,t)}_{\frac{\partial^2 u}{\partial x^2}}\, e^{-isx}\, dx$$

$$= \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 \mathcal{F}[u]$$

Get $\quad \boxed{\frac{\partial \hat{u}}{\partial t}(s,t) = -s^2\, \hat{u}(s,t)}$ ← An "ODE" with a parameter s floating along

$$\hat{u}(s,t) = \underbrace{C(s)}\, e^{-s^2 t}$$

constant might depend on s.
Also, ok for $C(s)$ to be complex #.

Next step: Get $C(s)$ from initial condition.

$$\underbrace{\hat{u}(s,0)}_{= C(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{f(x)}\, e^{-isx}\, dx$$

Aha! $$C(s) = \hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(-is)} e^{-isx} \right]_{x=-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-is}}{-is} - \frac{e^{is}}{(-is)} \right]$$

$$= \frac{1 \cdot 2}{\sqrt{2\pi}} \left[\frac{e^{is} - e^{-is}}{2i} \right] \cdot \frac{1}{s}$$

$$= \sqrt{\frac{2}{\pi}} \, \frac{\sin s}{s}$$

So $$\hat{u}(s,t) = \sqrt{\frac{2}{\pi}} \, \frac{\sin s}{s} \, e^{-s^2 t}$$

Last step: Undo $\tilde{\mathcal{F}}$. Get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \, \frac{\sin s}{s} \, e^{-s^2 t} \right) e^{isx}\, ds$$

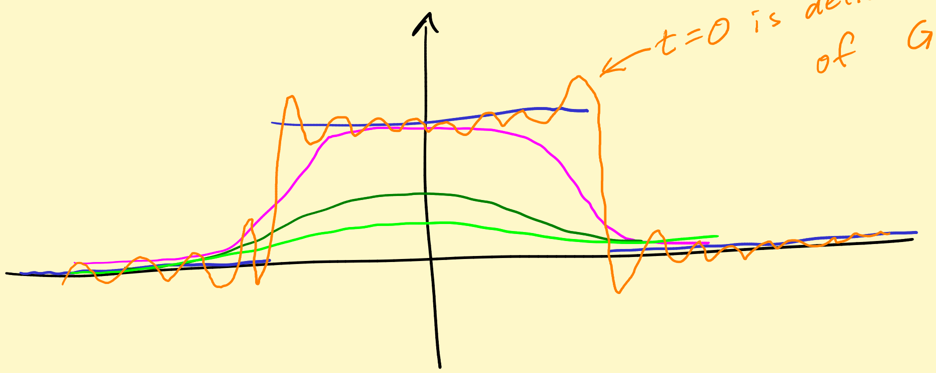
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \underset{\text{even}}{\frac{\sin s}{s}} \; \underset{\text{even}}{e^{-s^2 t}} \left(\underset{\text{even}}{\cos sx} + i \underset{\text{odd}}{\sin sx} \right) ds$$

Symmetric interval: $\int_{-\infty}^{\infty} = \lim_{A\to\infty} \int_{-A}^{A}$

(even)(even)(odd) = odd   So imaginary part of $\int_{-\infty}^{\infty} = 0$.

sol^n
$$\boxed{u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-s^2 t} \cos sx \, ds}$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin s}{s} e^{-s^2 t} \cos sx \, ds$$



As soon as $t>0$, solution becomes C^∞-smooth and integrals converge rapidly. Heat dissipates quickly and $u(x,t) \longrightarrow 0$ uniformly in x as $t \to \infty$.