Lecture 29 The Heisenberg uncertainty principle

HWK 7 due Thurs

Can't know both position and momentum of a particle.

Math version f and $\hat{f}$ cannot both have compact support.

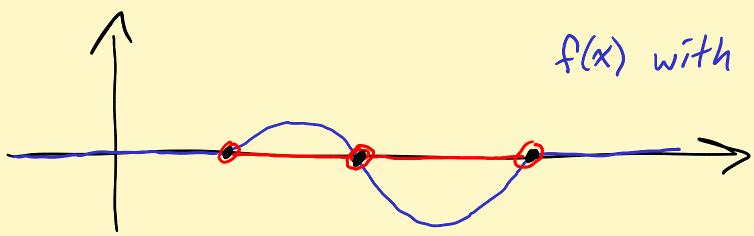
Support of f = $\operatorname{Supp} f$ = closure of the set of pts x where $f(x) \neq 0$.

$$= \overline{\{x : f(x) \neq 0\}}$$

$\operatorname{Supp} f$ = closure means it is closed.

Compact in $\mathbb{R}$ <=> closed and bounded.

Closure of a set $S \subset \mathbb{R}$ = $S \cup \{\text{all limit pts of } S\}$.



EX $\overline{[0,1)} = [0,1]$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Z}} = \mathbb{Z}$

Topology: S is closed <=> $\mathbb{R} - S$ is open

Thm Suppose f is C^1-smooth with compact support. Then $\hat{f}$ can't have compact support (unless $f \equiv 0$).

Case $\operatorname{Supp} f \subset (-\pi, \pi)$.

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

Hmmm. $\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, e^{-isx}\, dx$

$= 0 \quad \text{if } |s| > M \quad \text{if } \hat{f} \text{ has cmpct supp} \subset (-M, M)$

Aha! See Fourier series coeff

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$= 0$ when $|n| > M$.

Whoa! Fourier series for f on $(-\pi, \pi)$ is a trig poly!

Because f is C^1 smooth, Fourier series for f converges to f on $(-\pi, \pi)$. So

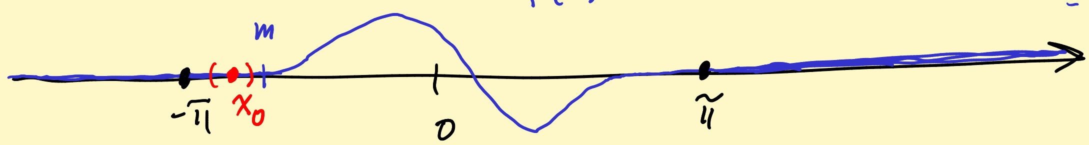
$$f(x) = \sum_{|n| \le M} c_n e^{inx}$$

Supp $f \subset (-\pi, \pi)$

$m = \min\{x \,;\, x \in \text{Supp} f\}$

$-\pi < m$

$f(x)$



Hmmm. Trig poly $\equiv 0$ in an interval $(x_0 - \delta, x_0 + \delta)$.

<u>Claim</u> This will imply that Trig poly $\equiv 0$ on $(-\pi, \pi)$

$\Longrightarrow f \equiv 0$ on $(-\pi, \pi)$ too

and so $f \equiv 0$ on $\mathbb{R}$. ✓

Taylor series fact!

$$\underbrace{f(x)}_{\text{real valued}} = \sum_{|n|\le M} c_n e^{inx} = \sum_{|n|\le M} (a_n + i b_n)\left[\cos nx + i \sin nx\right]$$

$$= \underbrace{\sum_{|n|\le M} (a_n \cos nx - b_n \sin nx)}_{= f(x) \text{ on } (-\pi,\pi)} + i \underbrace{\left(\text{who cares}\right)}_{\text{must } = 0}$$

$$\left.\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \end{aligned}\right\} \quad \text{radius of conv} = \infty$$

Hmmm. $\cos nx$, $\sin nx$ have power series expansions about $x = x_0$ with radius of conv $= \infty$ too.

So $$f(x) = \underbrace{\sum_{n=0}^{\infty} A_n (x - x_0)^n}_{\text{R.o.C.} = \infty} \quad \text{on } (-\pi, \pi)$$

Taylor's formula: $$A_n = \frac{f^{(n)}(x_0)}{n!} \leftarrow = 0 \text{ for all } n.$$

$f \equiv 0$ on $(-\pi, \pi)$. ✓

<u>Case</u> Supp $f \subset (-L, L)$. Know Fourier series on $(-L, L)$: $$f(x) = \sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$ where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx$$

$$= \frac{1}{2L} \int_{-\infty}^{\infty} f(x)\, e^{-i\frac{n\pi}{L}x}\, dx = 0$$

if $\left|\frac{n\pi}{L}\right| > M$

when $\operatorname{supp} \hat{f} \subset (-M, M)$.

Exact same argument works.

Improved version Can assume f is merely piecewise continuous and has compact support.

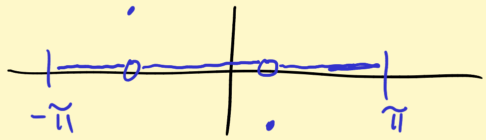
Sticking point: Fourier series for f is a trig poly.

Must f = trig poly on $(-\pi, \pi)$?

Yes!

Recall: f piecewise cont. If $\hat{f}(n) = 0$ for all n, then $f(x) = 0$ at pts x where f is continuous.

Parseval's:
$$\underbrace{\sum_{-\infty}^{\infty} |\hat{f}(n)|^2}_{=0} = \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} |f(x)|^2\, dx}_{>0 \text{ if } f(x) \neq 0 \text{ at cts pt } x.}$$



Aha! Fourier series for $f =$ trig poly $\sum_{|n| \le N} A_n e^{inx}$

Fourier series for $\underbrace{\left(f - \sum_{|n| \le N} A_n e^{inx}\right)}_{\substack{=0 \text{ where} \\ \text{continuous}}}$ is the zero F.S. !

See $f \equiv \sum_{|n| \le M} A_n e^{inx}$ on $(-\pi, \pi)$ except maybe at finitely many isolated pts. Change values there!

Now we can do the power series argument. ✓

Me : Fejér's $\Rightarrow$ Parseval's $\Rightarrow$ Uniqueness of Fourier coeff

Stein: Fejér's $\Rightarrow$ $\underbrace{\text{Uniqueness of Fourier coeff}}_{\substack{\text{made a "good kernel"} \\ \text{out of trig polys}}}$ $\Rightarrow$ Parseval's