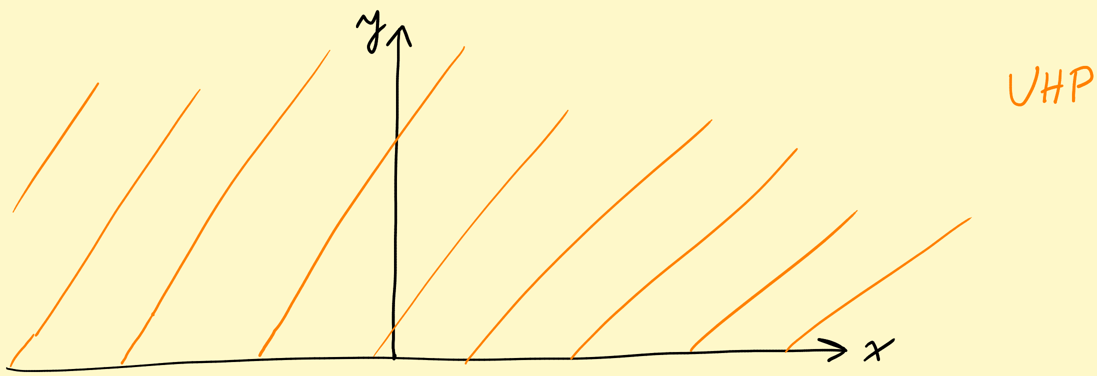
Lecture 30 Solving the Dirichlet problem on the upper half plane

HWK 8 due Thurs



Given continuous fcn $f(x)$ on $\mathbb{R}$ with $\lim_{x \to \pm\infty} f(x) = 0$.

Find a harmonic fcn u on the UHP that is continuous up to $\mathbb{R}$ with $u(x,0) = f(x)$.

Solution Steady state heat problem $c^2 \Delta u = \frac{\partial u}{\partial t} \equiv 0$ at steady state.

Laplace eqn $\Delta u \equiv 0$ in UHP.

Try $u(x,y) = X(x)\,Y(y)$

want

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$$

Case $\lambda = k^2 > 0$ $(k > 0)$ $\quad r^2 - k^2 = 0 \quad r = \pm k.$

$$\mathcal{X}(x) = c_1 e^{kx} + c_2 e^{-kx}$$

$\to \infty$ as $x \to \infty$

$\to \infty$ as $x \to -\infty$

Ouch! We don't want sol^{n}s $\to \infty$.

Case $\lambda = 0$ $\quad \mathcal{X}'' = 0$. $\quad$ Sol^n $\mathcal{X}(x) = Ax + B$

too hot as $x \to \infty$

Get $\mathcal{X}(x) = 1$

Case $\lambda = -k^2 < 0$ $\quad (k > 0)$ $\qquad r^2 - k^2 = 0 \quad r = \pm ki$

$$\mathcal{X}(x) = c_1 \cos kx + c_2 \sin kx$$

More daring today!

$$\mathcal{X}(x) = c_1 e^{ikx} + c_2 e^{-ikx}$$

Get $\mathcal{X}(x) = e^{isx} \qquad s = \pm k$.

Next, get $\mathcal{Y}$ fcns

Case $\lambda = 0$ $\qquad \mathcal{Y}(y) = 1$

Case $\lambda = -k^2$ $\qquad \mathcal{Y}'' - k^2 \mathcal{Y} = 0 \qquad r^2 - k^2 = 0 \quad r = \pm k$

$$\mathcal{Y}(y) = e^{-ky}, \ e^{ky}$$

too hot!

Found $\quad u(x,y) = X(x)\,Y(y) = e^{isx} e^{-|s|y} \qquad (s=\pm k)$

Hmmm. Problem is <u>linear</u>. So

$$\sum c_n e^{is_n x} e^{-|s_n| y} \quad \text{is a sol}^{n} \text{ too.}$$

Aha! Reminds me of a Riemann sum for $\int_{-\infty}^{\infty} \sim\!\sim\!\sim \, ds$

Try $\quad u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{isx} e^{-|s|y} \, ds$

If we assume $\int$ converges nicely, can take Δ under $\int$ to see that u is harmonic in UHP.

Last thing: $\quad$ Want $u(x,0) = f(x)$

$$\boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{isx} \, ds \overset{\text{want}}{=} f(x)}$$

$$\mathcal{F}^{-1}[g] = f$$

Aha! Need $g = \hat{f}$.

<u>Sol^{n}</u> $\quad u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx} e^{-|s|y} \, ds$

Poisson kernel for UHP :

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt \right) e^{isx}\, e^{-|s|y}\, ds$$

$$\underset{\text{Fubini}}{\uparrow}{=} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \underbrace{\left[\int_{-\infty}^{\infty} e^{is(x-t)-|s|y}\, ds \right]}_{P(x-t,\,y)} dt$$

Compute P $\qquad P(x-t,y) = \int_{-\infty}^{0} + \int_{0}^{\infty}$

$$= \underset{\substack{\uparrow \\ s<0}}{\int_{-\infty}^{0}} \qquad + \qquad \underset{\substack{\uparrow \\ s>0}}{\int_{0}^{\infty}} \qquad ds$$

$e^{-|s|y} = e^{sy}$

$\vdots$

$$= \frac{1}{i(x-t)+y}$$

$$\frac{1}{i(x-t)-y} \cdot e^{s[i(x-t)-y]} \Big|_{0}^{\infty}$$

$$0 - \frac{1}{i(x-t)-y}$$

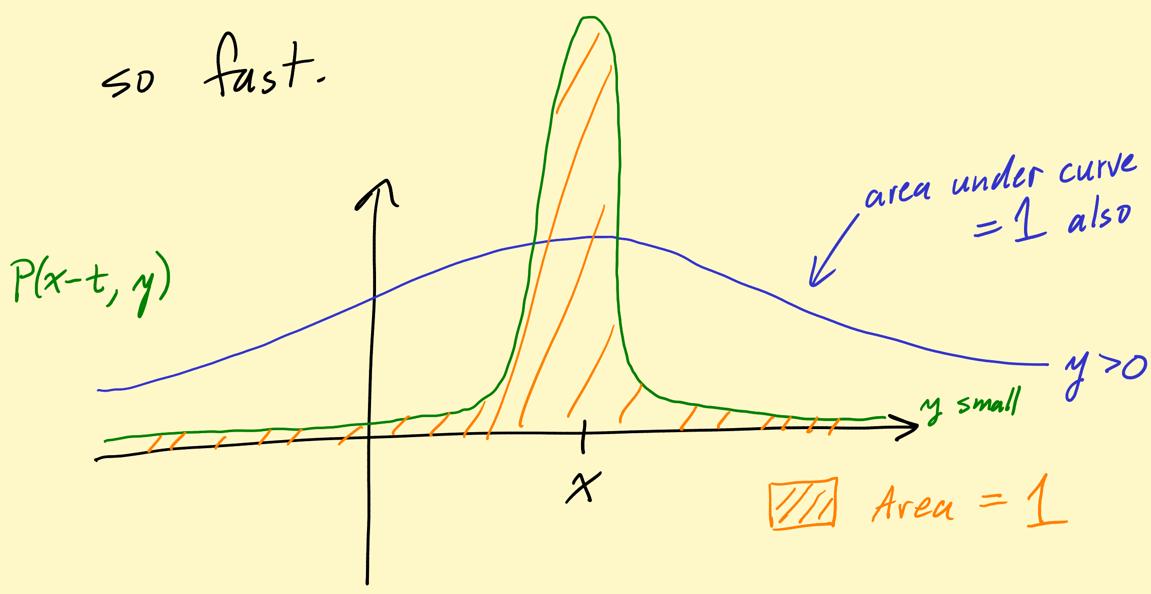
because $e^{-sy} \to 0$ as $s \to \infty$.

Add up: $\underbrace{P(x-t,y)}_{\text{Poisson kernel for UHP!}} = \dfrac{2y}{(x-t)^2+y^2}$

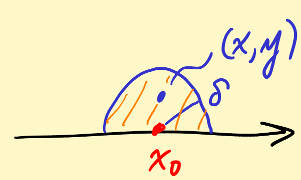
$$\boxed{u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{y}{(x-t)^2+y^2}\,dt}$$

$P(x-t,y)$ is harmonic in (x,y) on UHP via Δ under $\int$ which can be shown can be done because kernel $\to 0$ as $t \to \pm\infty$ so fast.

Good kernel!



Good kernel argument shows $u(x,y) \to f(x_0)$ as $(x,y) \to (x_0, 0)$ from the UHP



$$\left| e^{s\left[i(x-t)-y\right]} \right|$$

$$= \left| e^{i\left[s(x-t)\right]-sy} \right|$$

$$= \underbrace{\underbrace{\left| e^{i\left[s(x-t)\right]} \right|}_{=1} \cdot \underbrace{\left| e^{-sy} \right|}_{e^{-sy}}}_{\to 0 \text{ as } s \to \infty.}$$

$y>0$ UHP ✓