Lecture 31 Integrable functions on $\mathbb{R}$

Suppose f is continuous on $[a,b]$. Then f is integrable on $[a,b]$

Calculus facts A series $\sum_{n=1}^{\infty} u_n$ converges absolutely if $\sum_{n=1}^{\infty} |u_n| < \infty$. Lemma: A series that converges absolutely converges.

Why $\sum_{n=1}^{N} |u_n|$ is non-decreasing. If it's bounded above, it has a limit L.

If $\sum u_n$ is absolutely convergent, then

$S_N = \sum_{n=1}^{N} u_n$ is a Cauchy seq, and so it converges. (Completeness of $\mathbb{R}$.)

Calculus fact $\sum u_n$ converges $\Rightarrow$ $u_n \to 0$ as $n \to \infty$.

why $S_N = \sum_{1}^{N} u_n$

$$S_N - S_{N-1} = u_N$$

$$\underbrace{L - L}_{0}$$

Integrals f continuous on $\mathbb{R}$.

$\int_0^B |f|\,dx$ is non-decreasing in B.

Either $\int_0^B |f|\,dx \to +\infty$ as $B \to \infty$, or

it is <u>bounded</u> in B. $L = \sup\{\int_0^B |f|\,dx : B \geq 0\}$

= "the least upper bound".

Then $L = \lim_{B\to\infty} \int_0^B |f|\,dx$.

<u>Defn</u> If $L < \infty$, we say f is <u>absolutely</u> integrable on $[0,\infty)$ and we write

$$\int_0^\infty |f|\,dx = \lim_{B\to\infty} \int_0^B |f|\,dx.$$

<u>Fact</u> If f is absolutely integrable on $[0,\infty)$, then

$$\lim_{B\to\infty} \int_B^\infty |f|\,dx = 0.$$

<u>Why</u>: $$\underbrace{\int_0^\infty |f|\,dx}_{L} = \underbrace{\int_0^B |f|\,dx}_{\nearrow L} + \underbrace{\int_B^\infty |f|\,dx}_{\searrow 0}$$

$B \to \infty$:

If f is absolutely integrable, then $\lim_{B\to\infty} \int_0^B f\,dx$ exists and

$$\left| \int_B^\infty f \, dx \right| \le \int_B^\infty |f| \, dx .$$

HWK 7 : Problem 1 Prove Riemann-Lebesgue lemma for $\mathcal{C}^1$-smooth absolutely integrable fcns f on $\mathbb{R}$.

Note: f Absolutely int. on $\mathbb{R}$ means it is on $(-\infty, 0]$ and $[0, \infty)$.

$$\int_{-\infty}^{\infty} f \, dx = \int_{-\infty}^{0} f \, dx + \int_{0}^{\infty} f \, dx$$

$$= \lim_{\substack{A \to -\infty \\ B \to \infty}} \int_A^B f \, dx$$

R-L lemma:

$$\int_{-\infty}^{\infty} f(x) \underbrace{\sin wx}_{\substack{\text{or} \\ \cos wx \\ \text{or} \\ e^{iwx}}} \, dx \longrightarrow 0 \quad \text{as } w \to \infty.$$

Let $\varepsilon > 0$. Then, there is an N such that

$$\int_N^\infty |f| \, dx < \frac{\varepsilon}{3}$$

$$\int_{-\infty}^{-N} |f| \, dx < \frac{\varepsilon}{3}$$

(because these $\to 0$ as $N \to \infty$)

Important that

$$\left|\int_N^\infty f\ \sin wx\, dx\right| \leq \int_N^\infty |f|\ \underbrace{|\sin wx|}_{\leq 1}\, dx$$

$$\leq \int_N^\infty |f|\, dx < \frac{\varepsilon}{3}.$$

Same for $(-\infty, -N]$ side.

Fix N. Use integration by parts to bound

$$\left|\int_{-N}^N \underbrace{f(x)}_{u}\ \underbrace{\sin wx\ dx}_{dv}\right| \leq \frac{(\text{constant})}{w}$$

$V = -\frac{1}{w}\cos wx$

Use: f, f' continuous on $[-N, N] \Rightarrow$ both bounded there.

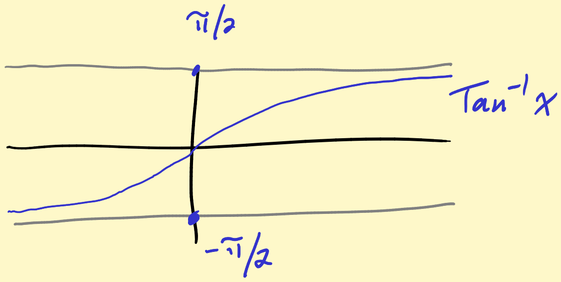
Similar idea for HWK 8, prob 1.

Stein p. 134–144 demonstrates the Fourier inversion formula!

$\mathcal{M}(\mathbb{R})$ = "functions of moderate decrease on $\mathbb{R}$"

$= \{ f : f$ continuous on $\mathbb{R}$ such that $|f(x)| \leq \frac{c}{1+x^2}$ when $|x| > M$ for some $c > 0$, $M > 0 \}$

Note $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{B\to\infty} \operatorname{Tan}^{-1}x\Big|_0^B = \frac{\pi}{2} - 0 = \frac{\pi}{2}$



Basic facts

1) $f \longmapsto \displaystyle\int_{-\infty}^{\infty} f\,dx$ is a linear operator $\mathcal{M}(\mathbb{R}) \longrightarrow \mathbb{R}$

2) $\displaystyle\int_{-\infty}^{\infty} f(\underbrace{x+h}_{u})\,dx = \int_{-\infty}^{\infty} f(x)\,dx$ Change of variable formula.

$du = dx$

3) $\displaystyle\int_{-\infty}^{\infty} f(\underbrace{kx}_{u})\,\underbrace{dx}_{\frac{1}{k}du} = \frac{1}{k}\int_{-\infty}^{\infty} f(x)\,dx$ ✓

$u = kx$

$du = k\,dx$

4) $\displaystyle\int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx \longrightarrow 0$ as $h \to 0$.

Pf of 4 : Given $\varepsilon > 0$. $\exists R > 0$ such that

$\displaystyle\int_{R-1}^{\infty} |f|\,dx < \frac{\varepsilon}{4}$ and

$$\int_{-\infty}^{-R+1} |f|\,dx < \frac{\varepsilon}{4}.$$

Assume $|h| < 1$. Now

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx = \int_{-\infty}^{-R} + \int_{-R}^{R} + \int_{R}^{\infty}$$

use uniform continuity of f on $[-R-1, R+1]$ to make $|f(x+h)-f(x)| < \frac{\varepsilon/4}{2R}$ on $[-R,R]$ when $|h| < \delta < 1$.

Other two are easy to bound:

$$\int_{R}^{\infty} |f(x+h) - f(x)|\,dx$$

$$\leq \underbrace{\int_{R}^{\infty} |f(x+h)|\,dx}_{< \int_{R-1}^{\infty} |f|\,dx \; < \frac{\varepsilon}{4}} + \underbrace{\int_{R}^{\infty} |f|\,dx}_{< \int_{R-1}^{\infty} |f|\,dx \; < \frac{\varepsilon}{4}}$$

Replace 4's by 5!

Google "Fourier transform". Wikipedia

1) $$\mathcal{F}_1[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi s x}\,dx$$

$$\mathcal{F}_1^{-1} = \int_{-\infty}^{\infty} \hat{f}_1(s)\, e^{i2\pi s x}\,ds$$

Stein prefers this one for theory

2) $$\tilde{\mathcal{F}}_2\left[f(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

Good for solving PDE's

$$\tilde{\mathcal{F}}_2^{-1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_2(s)\, e^{isx}\, ds$$

3) $$\tilde{\mathcal{F}}_3\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$\tilde{\mathcal{F}}_3^{-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_3(s)\, e^{isx}\, ds$$