Lecture 32 Fourier transform on Schwartz functions

HWK 8 due Thurs

1) $\tilde{\mathcal{F}}_1[f] = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x s}\, dx$ | $\tilde{\mathcal{F}}_1^{-1} = \int_{-\infty}^{\infty} \hat{f}_1(s) e^{i2\pi s x}\, ds$ — good for theory

2) $\tilde{\mathcal{F}}_2[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx$ | $\tilde{\mathcal{F}}_2^{-1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_2(s) e^{isx}\, ds$ — solving diff equs

3) $\tilde{\mathcal{F}}_3[f] = \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx$ | $\tilde{\mathcal{F}}_3^{-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_3(s) e^{isx}\, ds$ — happens in books

Version $\tilde{\mathcal{F}}_1$ is good for theory because

1) $$\tilde{\mathcal{F}}_1\left[e^{-\pi x^2}\right] = e^{-\pi s^2}$$

2) Planchere(l) identity: $\|f\|^2 = \|\hat{f}\|^2$ ← no constants involving π!

$$\|f\| = L^2\text{-norm} = \left(\int_{-\infty}^{\infty} |f|^2\, dx\right)^{1/2}$$

On Final exam, formula we will use is $\tilde{\mathcal{F}}_2$.

Schwarz ← German, complex analysis : Schwarz lemma

Schwartz ← French, Laurent Schwartz : Tempered distributions

Def^n $\mathcal{S}$ = "space of Schwartz fcns" on $\mathbb{R}$

is the space of "rapidly decreasing fcns at infinity"

= C^∞-smooth fcns $f(x)$ such that $\lim_{x\to\pm\infty} |f^{(n)}(x)\, x^m| = 0$.

for all $n, m = 0, 1, 2, \ldots$

EX $e^{-x^2} \in \mathcal{S}$

Exercise $f \in \mathcal{S} \iff f$ is C^∞-smooth and

$$\sup_{x \in \mathbb{R}} |f^{(n)}(x)\, x^m| < \infty$$

for each $n, m = 0, 1, 2, \ldots$

Easy facts $f \in \mathcal{S} \Rightarrow f' \in \mathcal{S}$

$f \in \mathcal{S} \Rightarrow xf \in \mathcal{S}$

Bigger fact $f \in \mathcal{S} \Rightarrow \hat{f} \in \mathcal{S}$

For theory days: $\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi sx}\, dx \quad (\mathcal{F}_1)$

Important properties of the Fourier transform on $\mathcal{S}$

Write $\hat{f}(s) = \mathcal{F}[f]$.

1) $f(x+h) \xrightarrow{\mathcal{F}} e^{2\pi i sh} \hat{f}(s)$

2) $f(x)\, e^{-i2\pi xh} \xrightarrow{\mathcal{F}} \hat{f}(s+h)$

3) $f(\delta x) \xrightarrow{\mathcal{F}} \frac{1}{\delta} \hat{f}(s/\delta)$

4) $f'(x) \xrightarrow{\mathcal{F}} 2\pi i s \hat{f}(s)$

5) $-2\pi i\, x f(x) \xrightarrow{\mathcal{F}} \hat{f}'(s)$

These can be used to see that $\mathcal{F}: \mathcal{S} \to \mathcal{S}$.

Pf 1-5: Write formula. Change variables. Use integration by parts.

Key property $\quad \mathcal{F}\left[e^{-\pi x^2}\right] = e^{-\pi s^2}$

Step 1 $\quad$ Claim: $\mathcal{F}\left[e^{-\pi x^2}\right]$ is real valued.

$$\int_{-\infty}^{\infty} \underset{\text{even}}{e^{-\pi x^2}}\ \underbrace{e^{-i2\pi s x}}_{[\underset{\text{even}}{\cos 2\pi s x} + i \underset{\text{odd}}{\sin 2\pi s x}]}\, dx = 2\int_0^{\infty} e^{-\pi x^2} \cos 2\pi s x \, dx \quad ✓$$

Step 2: Write $\quad F(s) = \widehat{\left(e^{-\pi x^2}\right)}$. $\quad$ Claim: $F'(s) = -2\pi s\, F(s)$

$$F'(s) = \frac{d}{ds} \int_{-\infty}^{\infty} e^{-\pi x^2}\, e^{-i2\pi s x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} \Big[\underbrace{-i2\pi x\, e^{-i2\pi s x}}_{\frac{\partial}{\partial s} e^{-i2\pi s x}}\Big]\, dx$$

$$= i \int_{-\infty}^{\infty} \Big[\underbrace{-2\pi x\, e^{-\pi x^2}}_{\varphi'(x)}\Big]\, e^{-i2\pi s x}\, dx$$

where $\varphi(x) = e^{-\pi x^2}$

$$= i \ \tilde{\mathcal{F}}[\varphi'] = i\left(i 2\pi s \underbrace{\hat{\varphi}(s)}_{=F(s)}\right)$$

$$= -2\pi s \, F(s)$$

Step 3: Solve ODE $\quad F'(s) + \underbrace{2\pi s}_{P(s)} F(s) = 0$

First order linear ODE in standard form.

Integrating factor $\quad u(s) = e^{\int P(s)\,ds} = e^{\pi s^2}.$

Multiply standard form ODE by u:

$$\underbrace{u\left[F' + 2\pi s F\right]}_{[uF]'} = 0$$

Conclude $\quad uF \equiv C$, a constant.

$$e^{\pi s^2} F(s) = C$$

$$F(s) = C e^{-\pi s^2}$$

Determine C by plugging in $s=0$:

$$F(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot \underbrace{1}_{= e^{-i2\pi 0 \cdot x}} \, dx = 1$$

$\boxed{C=1}$

Done!

Feeling: If Fourier inversion is true on Gaussian bump fcns, then its true more generally because you can approximate lots of types of fcns (especially fcns in $\mathcal{S}$) by linear combos of bumps.

Reason: Make a "good" kernel out of $\varphi(x) = e^{-\pi x^2}$.

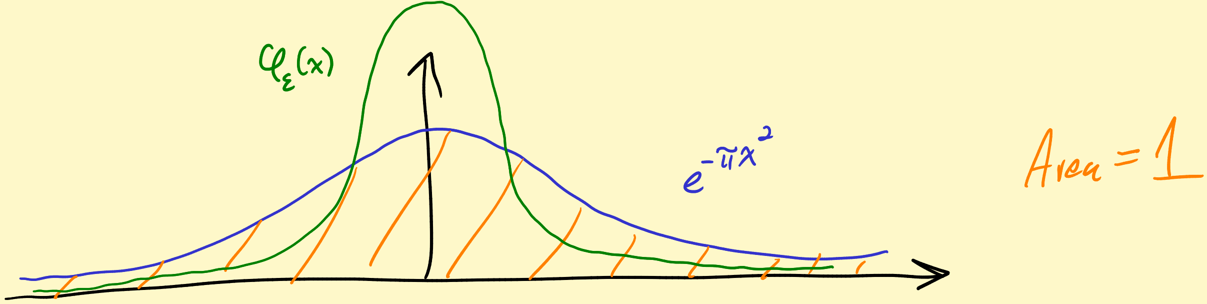
$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$. We showed that

$$f(x) \approx (\varphi_\varepsilon * f)(x) = \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(x-t)\, f(t)\, dt}_{\approx \text{ Riemann sum}}$$

$$\approx \sum_{1}^{N} \underbrace{\varphi_\varepsilon(x - t_i)}_{\substack{\text{bump fcn} \\ \text{centered at } t_i}} f(t_i)\, \Delta t$$

Important lemma: $f \in \mathcal{S}$. Then $\varphi_\varepsilon * f \to f$ uniformly on $\mathbb{R}$ as $\varepsilon \searrow 0$.

Pf: Did this earlier in semester using "good" kernel argument.



Area under φ_ε is also $= 1$.

$$(\varphi_\varepsilon * f)(x) - f(x)$$

$$\int_{-\infty}^{\infty} f(x-t)\,\varphi_\varepsilon(t)\,dt - \int_{-\infty}^{\infty} f(x)\,\varphi_\varepsilon(t)\,dt$$

$$= \int_{-\infty}^{\infty} \underbrace{\big(f(x-t) - f(x)\big)}_{A}\ \underbrace{\varphi_\varepsilon(t)}_{B}\,dt$$

split into pieces where A small, B small.