Lecture 33 Fourier inversion

HWK 9 due TUESDAY, April 22

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi s x}\, dx \qquad f(x) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{i2\pi s x}\, ds$$

when $f \in \mathcal{S} : \{ f \in C^\infty(\mathbb{R}) : x^m f^{(n)}(x) \to 0 \text{ all } n,m = 0,1,2,3 \}$.

Key fact $\varphi(x) = e^{-\pi x^2}$, $\hat{\varphi}(s) = e^{-\pi s^2}$. $\tilde{\mathcal{F}}[\varphi] = \varphi$

Lemma $f \in \mathcal{S}$. Then $\varphi_\varepsilon * f \to f$ uniformly on $\mathbb{R}$.

where $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$

Pf: $(\varphi_\varepsilon \circ f)(x) - f(x) =$

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, f(x-t)\, dt - \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, f(x)\, dt =$$

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(t) \underbrace{\left[f(x-t) - f(x) \right]}_{\substack{\text{claim: } \to 0 \\ \text{unif in } x \text{ as} \\ t \to 0}} dt$$

why: $f(t) \to 0$ as $t \to \pm\infty$.

$\varepsilon > 0$. Can make $|f(x)| < \frac{\varepsilon}{2}$

when $|x| > B$. Use unif cont

of f on $[-B,B]$ to get $\delta > 0$

so that $|f(x-t) - f(x)| < \varepsilon$ when $|t| < \delta$ for all x.

Let $\varepsilon > 0$. Get δ to make $|f(x-t) - f(x)| < \frac{\varepsilon}{2}$ when $|t| < \delta$.

$$(\varphi_\varepsilon * f)(x) - f(x) = \underbrace{\int_{-\delta}^{\delta} \varphi_\varepsilon(t) \underbrace{\left[f(x-t) - f(x) \right]}_{\text{small}} dt}_{I_1} + \underbrace{\int_{|t|>\delta} \varphi_\varepsilon(t) \left[f(x-t) - f(t) \right] dt}_{I_2}$$

$$|I_1| \le \int_{-\delta}^{\delta} \underset{>0!}{\varphi_\varepsilon(t)} \underbrace{|f(x-t) - f(x)|}_{< \varepsilon/2} dt$$

$$\le \frac{\varepsilon}{2} \int_{-\delta}^{\delta} \varphi_\varepsilon(t)\, dt < \frac{\varepsilon}{2} \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, dt}_{=1} \qquad \frac{\varepsilon}{2} \checkmark$$

I_2: $f \in \mathcal{S} \Rightarrow f$ is <u>bounded</u>. Say $|f(x)| \le M$ for all x.

$$|I_2| \le \int_{|t|>\delta} \varphi_\varepsilon(t)\, 2M\, dt \longleftarrow$$

HWK 7 showed $\to 0$ as $\varepsilon \to 0$ for fixed δ.

make it $< \frac{\varepsilon}{2}$. $\checkmark$

<u>Lemma</u> $f, g \in \mathcal{S}$

$$\int_{-\infty}^{\infty} f(x) \hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(x) g(x)\, dx$$

Pf Fubinni's on $f(x)\,g(y)\,e^{-i2\pi xy} = H(x,y)$

$$\iint_{\mathbb{R}^2} H(x,y)\,dA = \int_{-\infty}^{\infty} f(x) \underbrace{\left(\int_{-\infty}^{\infty} g(y)\, e^{-i2\pi xy}\, dy \right)}_{\hat{g}(x)} dx$$

$$= \int_{-\infty}^{\infty} g(y) \underbrace{\left(\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi xy}\, dx \right)}_{\hat{f}(y)} dy$$

Done! y is just a "dummy variable". Change it to x in last line.

Pf of Fourier inversion Enough to check only for $x=0$!

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(s)\, \underbrace{e^{-i2\pi 0\cdot s}}_{1}\, ds$$

$$\boxed{f(0) = \int_{-\infty}^{\infty} \hat{f}(s)\, ds}$$

Trick $$\int_{-\infty}^{\infty} f(x)\, \underbrace{\varphi_\varepsilon(x)}_{\substack{\text{want} \\ \| \\ \hat{g}(x)}}\, dx = (\varphi_\varepsilon * f)(0) \longrightarrow f(0) \text{ as } \varepsilon \to 0.$$

Easy to guess: $$\widehat{\varphi(\varepsilon x)} = \frac{1}{\varepsilon}\hat{\varphi}(s/\varepsilon) = \varphi_\varepsilon(s)$$

Get $$\int_{-\infty}^{\infty} f(x)\, \hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(s)\, g(s)\, ds$$

$$f(0) \xleftarrow[\varepsilon \to 0]{\text{as}} \int_{-\infty}^{\infty} f(x)\, \varphi_\varepsilon(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(s)\, \varphi(\varepsilon s)\, ds$$

$$= \underbrace{\int_{-\infty}^{\infty} \hat{f}(s)\, e^{-\pi \varepsilon^2 s^2}\, ds}_{\text{HWK 8.}} \longrightarrow \int_{-\infty}^{\infty} \hat{f}(s)\, ds \text{ as } \varepsilon \to 0$$

Done! for $x = 0$.

To get for all x, key
$$\mathcal{F}\left[f(\cdot + h)\right] = e^{2\pi i s h}\hat{f}(s)$$

Let $F(y) = f(y + x)$

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(s)\,ds = \underbrace{\int_{-\infty}^{\infty} e^{2\pi i s x} \hat{f}(s)\,ds}_{\text{that's the Fourier inversion formula!}}$$

$h = x$

Facts $\tilde{\mathcal{F}} : \mathcal{S} \to \mathcal{S}$

We just showed $\tilde{\mathcal{F}}^{-1}$ is same kind of operator.

Conclude $\tilde{\mathcal{F}} : \mathcal{S} \xrightarrow[\text{onto}]{1-1} \mathcal{S}$.

Cor Plancherel identity: For $f \in \mathcal{S}$

$$\|f\| = \|\hat{f}\|$$

$\leftarrow$ L^2-norm on $\mathbb{R}$

$\left(\int_{-\infty}^{\infty} |f|^2\,dx\right)^{1/2}$

Pf: Know $\displaystyle\int_{-\infty}^{\infty} f(x)\,\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(x)\,g(x)\,dx$

Hmmm. Want $\hat{g}(x) = \overline{f(x)}$.

Can have that if $g(x) = \tilde{\mathcal{F}}^{-1}\left[\overline{f(x)}\right]$

$$= \int_{-\infty}^{\infty} \overline{f(s)}\, e^{i2\pi s x}\,ds$$

$$= \int_{-\infty}^{\infty} \overline{\overline{f(s)}\ \overline{e^{-i2\pi sx}}}\ ds$$

$$= \overline{\int_{-\infty}^{\infty} f(s)\, e^{-i2\pi sx}\, ds}$$

$$= \overline{\hat{f}(x)}$$

$$\int_{-\infty}^{\infty} f(x)\, \overline{f(x)}\ dx = \int_{-\infty}^{\infty} \hat{f}(x) \left[\overline{\hat{f}(x)} \right] dx$$

$$\int_{-\infty}^{\infty} |f|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}|^2\, dx \quad \checkmark$$

Take square roots.

Fourier transform is an isometry in the L^2-norm on $\mathcal{S}$.

Consequently, the right space to do Fourier transform is $L^2(\mathbb{R})$ with respect to the Lebesgue integral.

Next time: Hilbert space, L^2, and Fourier analysis.