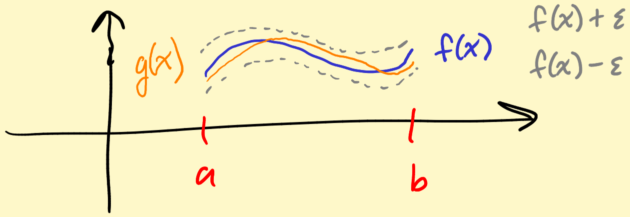
Lecture 34 Hilbert space, L^2, and the "right space" for Fourier analysis

Start thinking about functions as being "points" in a "space".

EX. $C[a,b]$ = continuous real valued fcns on $[a,b]$.

It is a "metric space" with distance fcn

$$d(f,g) = \sup_{[a,b]} |f-g|$$



$d(f,g) < \varepsilon$

Weierstraß Thm: Can uniformly approx a cont fcn by a poly on $[a,b]$.

So polynomials are dense in $C[a,b]$ the same way $\mathbb{Q}$ is dense in $\mathbb{R}$.

$C[a,b]$ is complete: Given a Cauchy sequence $\{f_n\}$ in distance sense: $d(f_n, f_m) < \varepsilon$ when $n, m > N$.

Then $\{f_n\}$ is uniformly Cauchy on $[a,b]$ $\Longrightarrow$ it converges uniformly to fcn f on $[a,b]$.

Know uniform limits of cont fcns are cont.

So f_n conv unif to f. $d(f_n, f) \to 0$ as $n \to \infty$. Unif Cauchy seq converges to f in $C[a,b]$.

First Hilbert spaces : $L^2[a,b]$, ℓ^2, $L^2(\mathbb{R})$.

$L^2[a,b]$ $\quad$ $\langle f, g\rangle = \int_a^b f(x)\,\overline{g(x)}\,dx$ $\quad$ $\|f\| = \left(\langle f,f\rangle\right)^{1/2}$

$\ell^2 = \left\{ (a_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$ $\quad$ $(a_n)\cdot(b_n) = \sum_{n=1}^{\infty} a_n \overline{b_n}$

$\|(a_n)\| = \left(\sum_1^{\infty} |a_n|^2\right)^{1/2}$

$L^2(\mathbb{R})$ $\quad$ $\langle f, g\rangle = \int_{-\infty}^{\infty}$ $\quad$ $\|f\| = \left(\langle f,f\rangle\right)^{1/2}$

Exciting thing $\quad$ Parseval's identity shows $\hat{f}(n)$ = Fourier coeff

is an isometry between $L^2[-\pi,\pi] \longleftrightarrow \ell^2$

(modulo multiplication by 2π someplace)

Need to know $L^2[-\pi,\pi]$ is complete. It is when

$L^2[-\pi,\pi]$ is the space of Lebesgue measurable

fcns that are square integrable with respect to the

Lebesgue integral.

Fact $\quad$ Isometry between Hilbert spaces implies inner products

are preserved : $\quad \|f\|^2 = \sum |\hat{f}(n)|^2$

$$\langle f, g\rangle = \sum \hat{f}(n)\overline{\hat{g}(n)}$$

Fourier transform $\quad \hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx$

$\mathcal{F} : \mathcal{S} \to \mathcal{S} \qquad$ Plancherel identity $\|f\| = \|\hat{f}\|$

Classic fact $\quad$ Can extend $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

as an "isometric isomorphism" $\leftarrow$ isometry

Sketch $\quad$ Step 1 $\quad$ Given $f \in L^2(\mathbb{R})$.

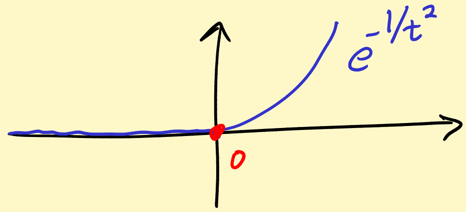
Approximate f in $L^2(\mathbb{R})$ by a seq of compactly supported fcns

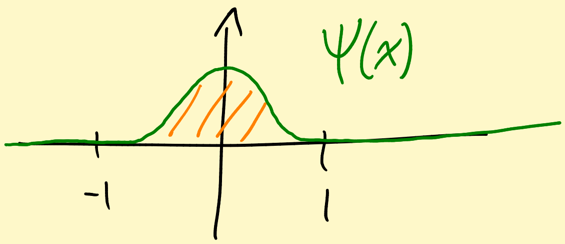
Easy! $\quad f_n = \chi_{[-n,n]} \cdot f \quad$ where $\chi_{[-n,n]} = \begin{cases} 1 & \text{in } [-n,n] \\ 0 & \text{outside} \end{cases}$

$f_n \to f$ in L^2 because $\displaystyle\int_{|x|>n} |f|^2\, dx \to 0$ as $n \to \infty$.

Step 2 $\quad$ Approx f_n by $\varphi_n \in C_0^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$.

Bump fcn trick: Need bump fcn $\psi \in C^\infty(\mathbb{R})$ with support in $(-1, 1)$





c = area under graph

$$\varphi(x) = \frac{1}{c}\psi(x)$$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$$

We showed that $g \longmapsto \varphi_\varepsilon * g$ is <u>smoothing</u> operator.

(same argument works in L^2.) (Lesson 20. page 4)

Aha! $\varphi_\varepsilon * f_n$ are C^∞ smooth with compact support.

<u>Step 3</u> Young's inequality shows that

$$\varphi_\varepsilon * f_n \xrightarrow{L^2} f_n \quad \text{as} \quad \varepsilon \to 0.$$

Have gotten a seq in $C_0^\infty(\mathbb{R}) \longrightarrow f$ in $L^2(\mathbb{R})$.

Call it $\{g_n\}$. Know $\|g_n - f\| \to 0$ as $n \to \infty$.

<u>Claim</u> $\{g_n\}$ is a Cauchy seq in $L^2(\mathbb{R})$.

MA 341 : $\|g_n - g_m\| = \|g_n - f + f - g_m\|$

$$\leq \|g_n - f\| + \|f - g_m\| \checkmark$$

Consequently, Plancherel's shows that

$$\|\hat{g}_n - \hat{g}_m\| = \|g_n - g_m\|$$

and so $\{\hat{g}_n\}$ is also a Cauchy seq.

$L^2(\mathbb{R})$ is complete. So $\hat{g}_n$ converges in L^2 to F.

F is the Fourier transform of f in L^2-sense

(Can show F is uniquely determined by this method.)

$$\begin{array}{ccc} \|g_n\| & = & \|\hat{g}_n\| \\ \downarrow & & \downarrow \\ \|f\| & = & \|F\| \end{array}$$

$\hat{f}$ for f in $L^2(\mathbb{R})$.

Why didn't we use $C_0^\infty(\mathbb{R})$ instead of $\mathcal{S}$?

Really needed fact $\underline{\mathcal{F}[e^{-\pi x^2}] = e^{-\pi x^2}}$

$$\langle f, \hat{g} \rangle = \langle \hat{f}, g \rangle \text{ on } \mathcal{S}$$

$$\mathcal{F}[f(\cdot + h)] = 2\pi i s h \, \mathcal{F}[f]$$

Wed: Fun, easy proof of the Fundamental Theorem of Algebra