Lecture 35 The Fundamental theorem of Algebra

No class Friday, no office hr Thurs.
video of Friday lecture in AFTERMATH

Complex polynomial: $P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$, a's $\in \mathbb{C}$.

Fund thm alg: A polynomial of degree $N \geq 1$ has a root in $\mathbb{C}$.

EX: $x^2 + 1 = (x - i)(x + i)$

Cor Polynomials factor: $P(z) = (z - r_1)^{m_1} (z - r_2)^{m_2} \cdots (z - r_j)^{m_j}$

$$\sum_{i=1}^{j} m_i = N.$$

Key fact Complex polys are complex differentiable (analytic).

Def^n: $f : \Omega \to \mathbb{C}$, Ω open set in complex plane.

$f(z)$ is $\mathbb{C}$-diff at $a \in \Omega$ if

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ exists. Call limit } f'(a).$$

EX $f(z) = z^2$ $\quad DQ = \dfrac{z^2 - a^2}{z - a} = \dfrac{(z-a)(z+a)}{z - a} = z + a$

$$\to a + a = 2a \quad \text{as} \quad z \to a.$$

Fact All freshman calculus facts hold true with same proofs

$\mathbb{C} \to \mathbb{C}$.

Reasons $\lim_{z \to a} F(z) = L$ means, given $\varepsilon > 0$, there is a

$\delta > 0$ such that $|F(z) - L| < \varepsilon$ when $|z - a| < \delta$, $z \neq a$.

In $\mathbb{R}$, $|\cdot|$ = absolute value. In $\mathbb{C}$, $|\cdot|$ = modulus.

Key: Δ ineq holds in $\mathbb{C}$: $|z+w| \le |z| + |w|$.

Product rule, quotient rule, etc. hold $\mathbb{C} \to \mathbb{C}$.

$\mathbb{C}$-diff: $1, \underbrace{z, z^2}_{z^3}, z^3, \ldots$ Poly's are $\mathbb{C}$-diff.

So rational fcns are too: $R(z) = \dfrac{P(z)}{Q(z)}$ polys

is $\mathbb{C}$-diff where $Q(z) \neq 0$.

Cauchy-Riemann eqns $f(x+iy) = u(x,y) + iv(x,y)$

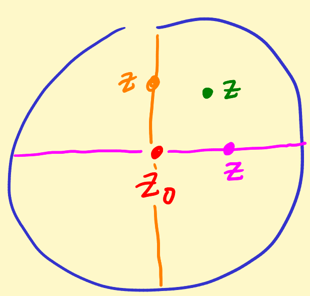
f $\mathbb{C}$-diff at $z_0 = x_0 + iy_0$, then

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \qquad \text{C-R eqns}$$

Suppose poly $P(z)$ of degree $N \ge 1$ has no zeroes in $\mathbb{C}$.

Then $\dfrac{1}{P(z)}$ would be $\mathbb{C}$-diff on $\mathbb{C}$ any number of times.

Why C-R eqns true



$$\frac{f(z) - f(z_0)}{z - z_0} \to \underset{\substack{\parallel \\ f'(z_0)}}{L}$$

horiz slide shows $\boxed{f'(z_0) = \dfrac{\partial u}{\partial x}(x_0,y_0) + i\dfrac{\partial v}{\partial x}(x_0,y_0)}$ Red box #1

vert slide shows $\boxed{f'(z_0) = \dfrac{\partial v}{\partial y}(x_0,y_0) - i\dfrac{\partial u}{\partial y}(x_0,y_0)}$ Red box #2

Equate red boxes to get C-R eqns.

Red box #2 $\quad \begin{cases} z = x_0 + iy \\ z_0 = x_0 + iy_0 \end{cases}$

$$DQ = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\left[u(x_0,y) + i v(x_0,y)\right] - \left[u(x_0,y_0) + i v(x_0,y_0)\right]}{i(y-y_0)}$$

$$= \frac{v(x_0,y) - v(x_0,y_0)}{y - y_0} - i \, \frac{u(x_0,y) - u(x_0,y_0)}{y - y_0} \quad \checkmark$$

$$\frac{1}{i}$$

Big fact If $f(z)$ is $\mathbb{C}$-diff and $f'(z)$ is too, then

$$f(x+iy) = u(x,y) + i\,v(x,y)$$

u, v are harmonic.

Fact Real and imaginary parts of analytic fcns are harmonic.

Why $\quad f = u + iv$

$$\left.\begin{array}{ll} \text{Red box \#1} & f' = u_x + i\,v_x \\ \text{Red box \#2} & f' = v_y - i\,u_y \end{array}\right\} = U + iV$$

U, V satisfy C-R eqns too!

$$\frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial u}{\partial x}}_{U}\Big) = \frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial v}{\partial x}}_{V}\Big) = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial v}{\partial y}}_{U}\Big) = \frac{\partial}{\partial y}\Big(\underbrace{-\frac{\partial u}{\partial y}}_{V}\Big)$$

$u_{xx} = -u_{yy}$ $\qquad \Delta u = u_{xx} + u_{yy} = 0$ ✓

Repeat for $\frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial u}{\partial x}}_{U}\Big) = -\frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial v}{\partial x}}_{V}\Big) = \frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial v}{\partial y}}_{U}\Big) = -\frac{\partial}{\partial x}\Big(\underbrace{-\frac{\partial u}{\partial y}}_{V}\Big)$

$-v_{xx} = v_{yy}$ $\qquad \Delta v = 0$ too.

<u>Proof of Fund Thm Alg</u> $\quad \frac{1}{P(z)} = u + iv$, u, v harmonic.

↑ no zeroes

harmonic fcns satisfy the ave property.

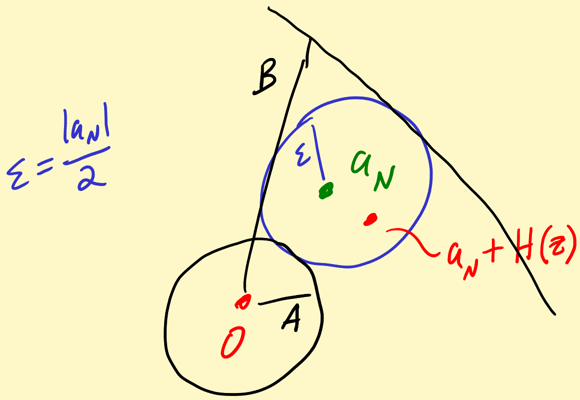
<u>Key</u>: $\frac{1}{P(0)} = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} u(R\cos\theta, R\sin\theta) + iv(R\cos\theta, R\sin\theta)\, d\theta}_{\to 0 \text{ as } R\to\infty.}$

↑ not zero

Contradiction!

Lemma $P(z) = a_N z^N + \cdots + a_1 z + a_0$

$$= z^N \left(a_N + \underbrace{\left(\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N} \right)}_{H(z) \to 0 \text{ as } |z| \to \infty.} \right)$$



$\exists\ R > 0$ such that

$|H(z)| < \frac{|a_N|}{2}$ when $|z| > R$.

$$\frac{P(z)}{z^N} = a_N + H(z)$$

Basic poly estimate: $\underbrace{A}_{\frac{|a_N|}{2}} |z|^N \le |P(z)| \le \underbrace{B}_{\frac{3|a_N|}{2}} |z|^N$ when $|z| > R$

So $$\underbrace{\frac{1}{B|z|^N} \le \left| \frac{1}{P(z)} \right| \le \frac{1}{A|z|^N}}_{\text{This shows } \frac{1}{P(z)} \to 0 \text{ very fast as } |z| \to \infty.} , \quad |z| > R$$

Average of a really small fcn is really small!

Can be made arbitrarily small.