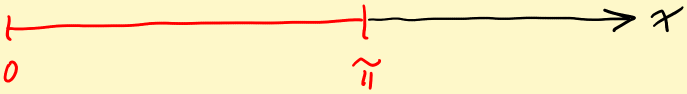
# Lecture 37 Generalized Fourier series

## Hot wire problem with one insulated end, other held at zero.

$u(x,t) = \text{temp}$




$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{PDE}$$

insulated end BC

$$\begin{cases} \dfrac{\partial u}{\partial x}(0,t) = 0 \\ u(\pi, t) = 0 \end{cases} \qquad \text{BC}$$

$\leftarrow$ homog boundary cond

$$u(x,0) = f(x) \qquad \text{IC}$$

Take $c=1$

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$X T' = X'' T$$

$$\frac{T'}{T} = \frac{X''}{X} = \lambda$$

$x=0$ insulated end:
No heat flow.
Kelvin's law:
(rate of heat flow)
= $-k$(Temp gradient)
Temp gradient $= \frac{\partial u}{\partial x}$

BC's: $\frac{\partial u}{\partial x}(0,t) = X'(0)\,T(t) \equiv 0$ for all $t$.

Don't want $T(t) \equiv 0$ because only get zero sol$^n$.

Need $X'(0) = 0$

$u(\pi,t) = X(\pi)\,T(t) \equiv 0$ for all $t$.

Need $X(\pi) = 0$.

$T$ problem: $\quad T' - \lambda T = 0$

Easy $T(t) = c\,e^{\lambda t}$. Take $c=1$.

$X$ problem: $$X'' - \lambda X = 0$$ $$\begin{cases} X'(0) = 0 \\ X(\pi) = 0 \end{cases}$$

Case $\lambda > 0$ Write $\lambda = k^2$ $(k>0)$. $r^2 - k^2 = 0$

$r = \pm k$

$$X(x) = c_1 e^{kx} + c_2 e^{-kx}$$

$$X'(x) = c_1 k e^{kx} - c_2 k e^{-kx}$$

want

$$X'(0) = c_1 k - c_2 k = 0 \qquad \boxed{c_2 = c_1}$$

So $X(x) = c_1 \left(e^{kx} + e^{-kx}\right)$

want

$$X(\pi) = c_1 \big(\underbrace{e^{k\pi} + e^{-k\pi}}_{>0}\big) = 0$$

Ouch! Get $c_1 = 0$. $X(x) \equiv 0$. Zero sol$^n$.

Case $\lambda = 0$ $X'' = 0$ $X(x) = Ax + B$

want

$X'(0) = A = 0$ So $A = 0$

So $X(x) \equiv B$.

want

$X(\pi) = B = 0$ So $B = 0$ too.

Zero sol$^n$ again

Case $\lambda < 0$ Write $\lambda = -k^2$ $(k>0)$ $r^2 + k^2 = 0$

$$X(x) = c_1 \cos kx + c_2 \sin kx$$

$$X'(x) = -c_1 k \sin kx + c_2 k \cos kx$$

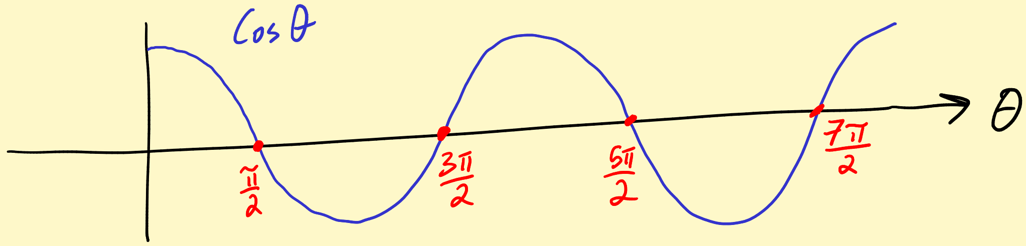
$$X'(0) = c_2 k \xleftarrow{\text{want}} = 0 \qquad \boxed{c_2 = 0}$$

So $X(x) = c_1 \cos kx$

Need $X(\pi) = c_1 \cos k\pi \overset{\text{want}}{=} 0$

Don't want $c_1 = 0$ too. Would only get zero sol$^n$.

Need $\boxed{\cos k\pi = 0}$




Need $k\pi = \frac{\pi}{2} + n\pi, \quad n = 0, 1, 2, \ldots$

$$k = \left(n + \tfrac{1}{2}\right) \qquad n = 0, 1, 2, \ldots$$

$$\lambda = -k^2 = -\left(n + \tfrac{1}{2}\right)^2$$

Get $X_n(x) = c_1 \cos\left(n + \frac{1}{2}\right)x \quad \leftarrow$ take $c_1 = 1$.

$$T_n(t) = e^{\lambda t} = e^{-\left(n+\frac{1}{2}\right)^2 t}$$

$$u_n(x,t) = X_n(x)\, T_n(t) = e^{-\left(n+\frac{1}{2}\right)^2 t} \cos\left(n + \tfrac{1}{2}\right)x$$

Finally, think about IC.

Try $\quad u(x,t) = \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2 t} \cos(n+\tfrac{1}{2})x$

IC $\quad u(x,0) = \underbrace{\sum_{n=0}^{\infty} A_n \cos(n+\tfrac{1}{2})x}_{\text{generalized Fourier series}} \overset{\text{want}}{=} f(x)$

Good news: Fcns you get from a Sturm-Liouville problem are guaranteed to be $\perp$.

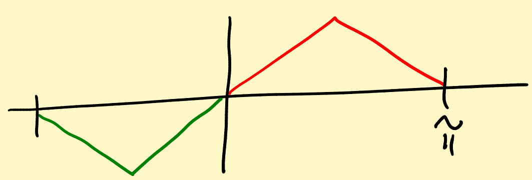
$$A_n \int_0^{\pi} \underbrace{\cos^2(n+\tfrac{1}{2})x}_{\frac{1}{2}\left(1+\cos 2(n+\frac{1}{2})x\right)}\,dx = \int_0^{\pi} f(x)\cos(n+\tfrac{1}{2})x\,dx$$

$A_n$'s determined by this formula.

<u>Question</u> Is $\{\cos(n+\frac{1}{2})x\}_{n=0}^{\infty}$ a <u>complete</u> orthogonal system?

[Is Bessel's ineq an equality: Parseval's identity.]

Sneaky things: $\{\sin nx\}_{n=1}^{\infty}$ on $[0,\pi]$




extend to be $2\pi$-periodic

Full Fourier series for this <u>odd</u> fcn has no cosine terms!

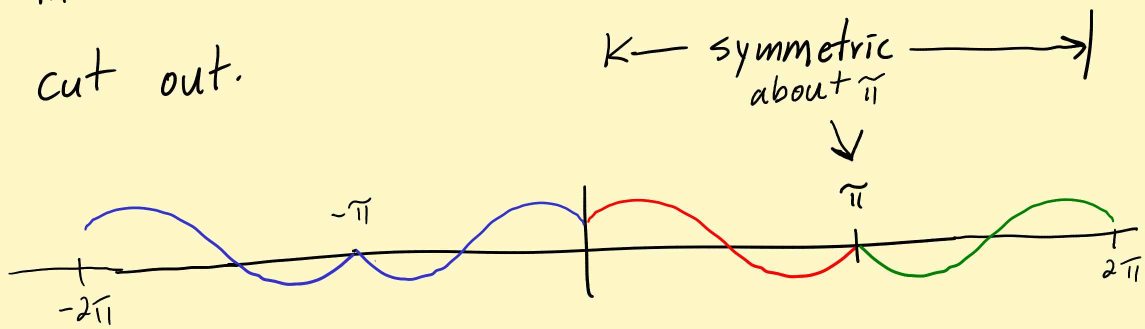
We showed $\{\sin nx\}_{n=1}^{\infty}$ are a complete $\perp$ system on $[0,\pi]$

Similary, by taking an <u>even</u> extension, also see

$\{1, \cos nx \,|_{n=1}^{\infty}\}$ are complete on $[0,\pi]$

<u>Hmmm</u> $\quad (n+\tfrac{1}{2})\,|_{n=0}^{\infty} \quad \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}$

Get desperate. Try various types of clever extensions to intervals to make various terms in full Fourier series cut out.




Take <u>even</u> extension

Then take $4\pi$-periodic extension

Does the full Fourier series on $[-2\pi, 2\pi]$ restricted to $[0,\pi]$ have just the right missing terms to see $\{\cos(n+\tfrac{1}{2})x\}_{n=0}^{\infty}$ is complete? (No! Better try next time.)

Otherwise, repeat the whole theory of Fourier series in the more general context of Sturm-Liouville problems.

Next time: Vibrating circular drum head. Bessel functions!