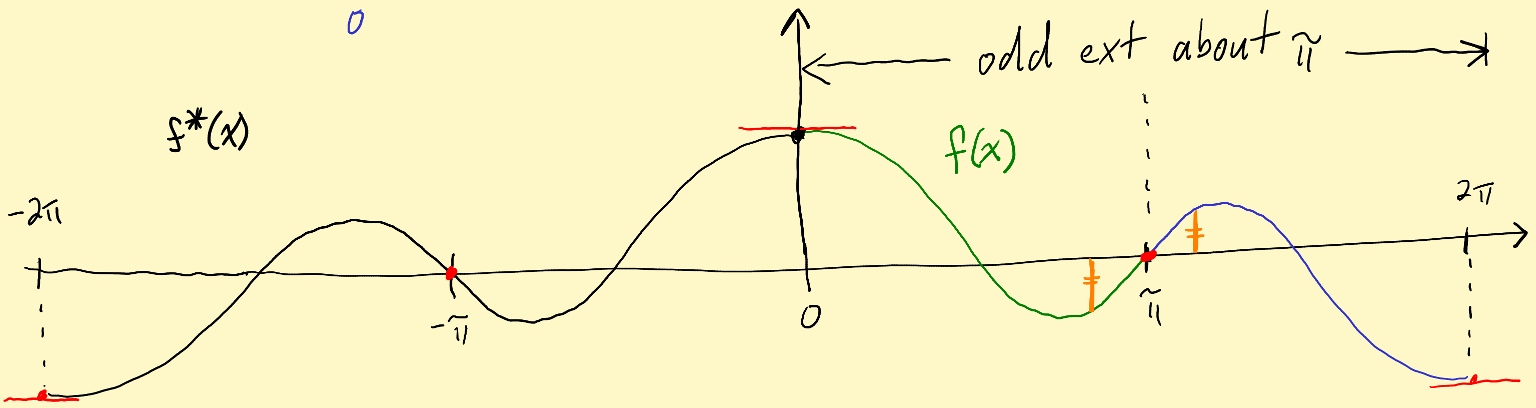
Lecture 38 Vibrating circular drum problem

HWK 9 due Friday

Loose end from last time. Want to show $f(x)=\sum_{n=0}^{\infty} A_n \cos(n+\tfrac{1}{2})x$

where $A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(n+\tfrac{1}{2})x\,dx.$ $f'(0)=0,\ f(\pi)=0$



Extend as an <u>even</u> fcn from $[0,2\pi]$ to $[-2\pi,2\pi]$.

Extend to $\mathbb{R}$ as a 4π-periodic fcn, Get $f^*(x)$.

$$f^*(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{(2\pi)}x + b_n \sin\frac{n\pi}{(2\pi)}x \qquad (L=2\pi)$$

b_n's $=0$ because f^* even.

$a_0 = 0$ because ave f^* on $[-2\pi,2\pi]$ is zero.

<u>Claim</u>: $a_n = 0$ for <u>even</u> n.

$n=2$: $$a_2 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} = \frac{1}{\pi}\int_0^{2\pi} \underbrace{\underbrace{f^*(x)}_{\substack{\text{odd}\\\text{about}\\x=\pi}}\ \underbrace{\cos x}_{\substack{\text{even}\\\text{about}\\x=\pi}}}_{\text{odd about } x=\pi}\,dx = 0$$

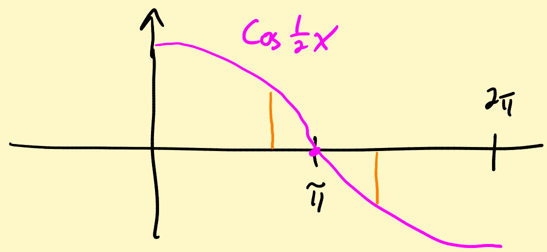
Same for $n=4,6,8,\ldots$

n odd: $$a_1 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} = \frac{1}{\pi}\int_0^{2\pi} \underbrace{f^*(x)}_{\text{odd}}\ \underbrace{\cos\tfrac{1}{2}x}_{\text{odd}}\,dx$$

about $x=\pi$ about $x=\pi$

even about $x=\pi$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos\tfrac{1}{2}x\,dx \quad \checkmark$$



Same for $\cos\frac{3}{2}x$, $\cos\frac{5}{2}x$, ---.

Aha! $\sum_{n=0}^{\infty} A_n \cos(n+\frac{1}{2})x$ is the full Fourier series for f^*

on $[-2\pi, 2\pi]$.

Remark This yields a Parseval's identity! $L=2\pi$

$$\underbrace{2\underset{\overset{\shortparallel}{0}}{a_0^2} + \sum_{n=1}^{\infty}(a_n^2 + \underset{\overset{\shortparallel}{0}}{b_n^2})}_{\sum_{n=0}^{\infty} A_n^2} = \underbrace{\frac{1}{\underbrace{(2\pi)}_{L}}\int_{-2\pi}^{2\pi} |f^*|^2\,dx}_{\frac{1}{\pi}\int_0^{2\pi}|f^*|^2\,dx}$$

$$\sum_{n=0}^{\infty} A_n^2 = \frac{1}{\pi}\int_0^{2\pi}|f^*|^2\,dx = \frac{2}{\pi}\int_0^{\pi}|f|^2\,dx$$

Works when f is merely continuous.

When f is C^1-smooth, $f'(0)=0$, $f(\pi)=0$, f^* is C^1-smooth on $\mathbb{R}$. Fourier series converges uniformly to f^* on $\mathbb{R}$, and so uniformly to f on $[0,\pi]$.

Another possible boundary condition

$\frac{\partial u}{\partial x}(0,t)=0$ ← Insulated end

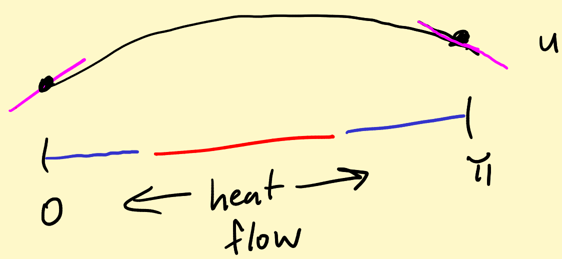
$u(0,t)$ ← homog bndry cond. (Refridgerated end)

Radiative boundary condition

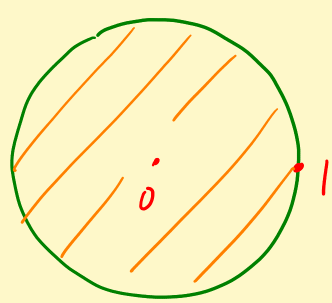
(Rate of heat loss) = (Rate of heat flow)

$$= \begin{cases} -K\,(\text{Temp gradient}) \\ (\text{const})\,(\text{Temp}) \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial x}(0,t) = c\,u(0,t) \quad c>0 & \text{(left end)} \\ \frac{\partial u}{\partial x}(\pi,t) = -c\,u(\pi,t) & \text{(right end)} \end{cases}$$



Vibrating circular membrane



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

BC $u \equiv 0$ on circle

Polar coord: $u(r,\theta,t)$ BC $u(1,\theta,t) \equiv 0$

IC: Initial conditions: $\left\{ \begin{array}{l} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{array} \right\}$ given

Take $c=1$. In polar coords, the Laplacian Δ is:

PDE $$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

Try $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

Plug into PDE $$R\Theta T'' \overset{\text{want}}{=} R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T$$

Divide by $R\Theta T$:

$$\underbrace{\frac{T''}{T}}_{\text{fcn of } t} = \underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{fcn of } r,\theta\ !} = \lambda$$

Get λ by holding $t=\frac{1}{2}$. RHS = const λ.

Let t go free. See LHS $\equiv \lambda$ too.

Get T-prob: $T'' - \lambda T = 0$. Easy

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda$$

Multiply by r^2: $\underbrace{r^2 \frac{R''}{R} + r\frac{R'}{R} - \lambda r^2}_{\text{fcn of } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{fcn of } \theta} = \mu$

Θ-problem: $\quad \Theta'' + \mu\Theta = 0$

no kinks or rips of drum head:

Need periodic boundary conditions.

Easy. We've done it.

R-problem: $\quad r^2 R'' + rR' - (\mu + \lambda r^2)R = 0$

Bessel equation!

Hidden BC: $R(0)$ needs to be finite.