Lecture 39 Bessel functions, Legendre's eqn

$$\tilde{\mathcal{F}}[f](s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx$$

or t ... t here

Could change x's to another "dummy variable" too.

$$\tilde{\mathcal{F}}^{-1}[f](s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

Notice that $\tilde{\mathcal{F}}^{-1}[f](-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(-s)x}\, dx = \tilde{\mathcal{F}}[f](s)$

Find the Fourier transform of $\frac{2}{\pi} \int_0^{\infty} \frac{\cos sx}{s^2+1}\, ds$

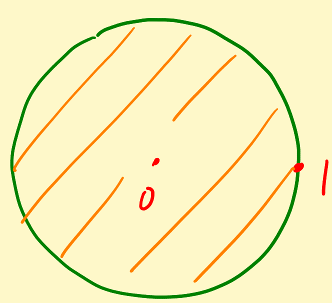
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx}{s^2+1}\, ds$$

Hmmm. $\cos sx = \mathrm{Re}\left[e^{-isx}\right]$

Play around : $= c\, \tilde{\mathcal{F}}[g](x) = G(x)$

Hmmm $\tilde{\mathcal{F}}[G](x) = \tilde{\mathcal{F}}^{-1}[G](-x)$. Aha! $\tilde{\mathcal{F}}^{-1}[\tilde{\mathcal{F}}] = id$

Vibrating circular membrane



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

BC $u \equiv 0$ on circle

Polar coord: $u(r,\theta,t)$ BC $u(1,\theta,t) \equiv 0$

IC: Initial conditions: $\left\{ \begin{array}{l} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{array} \right\}$ given

Take $c=1$. In polar coords, the Laplacian Δ is:

PDE $$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

Try $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$

Plug into PDE $$R\Theta T'' \overset{\text{want}}{=} R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T$$

Divide by $R\Theta T$:

$$\underset{\substack{\uparrow \\ \text{fcn of} \\ t}}{\frac{T''}{T}} = \underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{fcn of } r,\theta\ !} = \lambda$$

Get λ by holding $t=\frac{1}{2}$. RHS = const λ.

Let t go free. See LHS $\equiv \lambda$ too.

Get T-prob: $T'' - \lambda T = 0$. Easy

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda$$

Multiply by r^2 : $\underbrace{r^2\frac{R''}{R} + r\frac{R'}{R} - \lambda r^2}_{\text{fcn of } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{fcn of } \theta} = \mu$

Θ-problem : $\Theta'' + \mu\Theta = 0$

no kinks or rips of drum head : $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$

Need periodic boundary conditions.

Easy. We've done it. $\mu = n^2$, Θ's = Fourier basis fcns $1, \cos n\theta, \sin n\theta$

R-problem : $r^2 R'' + rR' - (n^2 + \lambda r^2)R = 0$

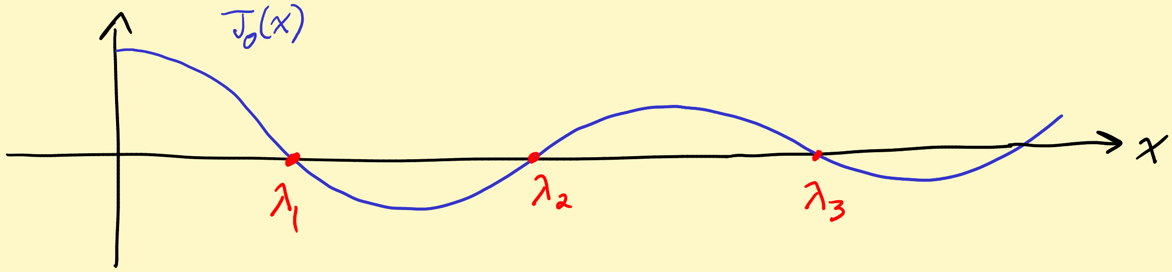
Bessel equation! of order n. (after scaling r variable)

Hidden BC : $R(0)$ needs to be finite.

<u>Interesting case</u> $n=0$. $J_0(x)$ is the Bessel fcn of the "first kind" of order 0.

$Y_0(x)$ is the Bessel fcn of the second kind. It blows up at $x=0$. We discard Y_0. [$c_1 J_0 + c_2 Y_0$ is the gen^l sol^n.]

$r=0$ is a "regular singular point" of the ODE.



$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

converges faster than series for e^x!

Radius of convergence $= \infty$.

Find it by plugging $\sum_{n=0}^{\infty} c_n x^n$ in ODE, collecting coeff and recursively solving for them.

Write λ_{om} for zeroes of J_0. $m=1,2,\ldots$

Boundary condition. $R_0(r) = J_0(kr)$ (k comes from the λ) $-k^2=\lambda$

Need $R_0(1) = J_0(k\cdot 1) = 0$ need

k's have to be the zeroes of the Bessel fcn!

Get bunch of sol^{n}s $R_{om}(r) = J_0(\lambda_{om} r)$

Get more from other n's: $R_{nm}(r) = J_n(\lambda_{nm} r)$

<u>Sturm-Liouville problem</u> R_{nm} are $\perp$ on $[0,1]$

with respect to a weight fcn r: $\langle u,v\rangle = \int_0^1 u(r)v(r)\, r\, dr$

Whole theory works just like Fourier series!

Sol^n

overtones!

$$u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \Big[A_{nm} J_n(\lambda_{nm} r) \cos n\theta \cos \lambda_{nm} t$$

$$+ B_{nm} J_n(\lambda_{nm} r) \sin n\theta \cos \lambda_{nm} t$$

$$+ C_{nm} J_n(\lambda_{nm} r) \cos n\theta \sin \lambda_{nm} t$$

$$+ D_{nm} J_n(\lambda_{nm} r) \sin n\theta \sin \lambda_{nm} t \Big]$$

Boyce & DiPrima : last chapters

What happens in 3-dimensions?

Heat eqn in a metal ball. Use spherical coords.

Separation of variables.

ODE for $R(\rho)$ is the Legendre eqn.

Get orthogonal fcns called the "Legendre polynomials"

Famous fact If f is $\mathcal{C}^1$-smooth on $[-\pi,\pi]$, then the Fourier coeff for f are absolutely sumable : $\sum_{-\infty}^{\infty} |c_n| < \infty$.

Hint: $f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$f' \sim \sum_{n=-\infty}^{\infty} in c_n e^{inx}$ f' continuous. Parseval's $\sum_{-\infty}^{\infty} n^2 |c_n|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |f'|^2 dx$

$$\left(|c_n| \right) = \left(n \tfrac{1}{n} |c_n| \right) \leftarrow \text{vector}$$

$$\sum_{n \neq 0} |c_n| = \left(\frac{1}{n} \right) \cdot \left(n|c_n| \right) \leq \left(2 \sum_{1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left(\text{Parseval's for } f' \right)^{1/2}$$

$\uparrow$

Toss the c_0 term!