Review 1   Final exam, Friday, May 9, 7-9 pm RHPH 164.

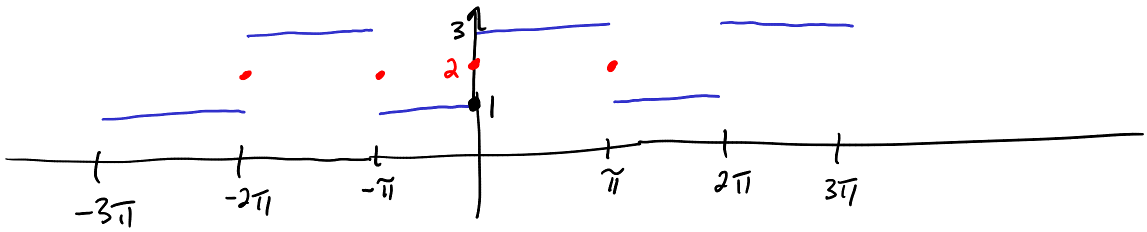
No class on Friday, May 2.   [2 standard sized sheets handwritten on both sides]

Formula sheet on back page of exam (see home page).

1. Let $f(x) = \begin{cases} 1 & -\pi < x < 0 \\ 3 & 0 < x < \pi. \end{cases}$, and let $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ $= s(x)$

be the Fourier series for f on $[-\pi, \pi]$. Evaluate

$$s(\pi) \qquad \sum_{n=0}^{\infty}(-1)^n a_n = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi \; + b_n \underbrace{\sin n\pi}_{=0}. \qquad = 2$$



2. If the Fourier sine Series of $f(x) = x^4$ on $[0, \pi]$ is $\sum_{n=1}^{\infty} b_n \sin nx$, evaluate the sum of the infinite series $\sum_{n=1}^{\infty} b_n^2$.

Parseval's: Piecewise continuous $f(x)$ on $[-\pi, \pi]$.

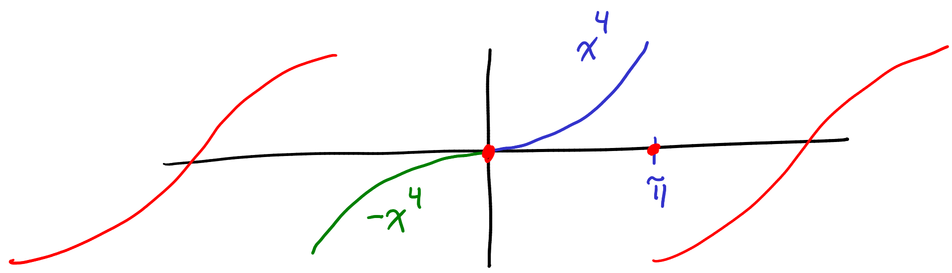
Full Fourier series $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 \, dx$$

Fourier sine series for $f(x) = x^4$ on $[0, \pi]$ =

Full Fourier s. for odd 2π-periodic extension has only sine terms.

f^*



$$s^*(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow x^4 \text{ on } [0, \pi]$$

$$s^*(x) \longrightarrow f^*(x) \text{ on } [-\pi, \pi].$$ (but $s^*(\pi) = 0$, not π^4.)

Claim: $$2 \cdot 0^2 + \sum_{n=1}^{\infty} (0^2 + b_n^2) = \sum_{n=1}^{\infty} b_n^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |f^*(x)|^2 \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x^4)^2 \, dx$$

$$= \frac{2}{\pi} \left(\frac{1}{9} \pi^9 \right)$$

3. Suppose $f(x) = 1 + \cos x + \sum_{n=1}^{\infty} \underbrace{\left|\frac{1}{n^2}\sin nx\right|}_{\leq M_n}$ on $[-\pi, \pi]$.

$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

integral test

$\left(= \frac{\pi^2}{6}\right)$

Explain why f is continuous. ← Weierstraß M-test → uniformly

Evaluate $\int_{-\pi}^{\pi} f(x)\cos 5x\,dx$ and $\int_{-\pi}^{\pi} f(x)\sin 5x\,dx$.

Uniform limit of cont. fcns is cont.

$M_n = \frac{1}{n^2}$

$\sin 5x \perp$ all terms in sum <u>except</u> $\sin 5x$ term

$$\int_{-\pi}^{\pi} f(x)\sin 5x\,dx = 0 + 0 + \cdots + 0 + \int_{-\pi}^{\pi} \frac{1}{5^2}\sin^2 5x\,dx + 0 + \cdots$$

$$= \frac{1}{25}\int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 10x)\,dx$$

$$\int_{-\pi}^{\pi} f(x)\cos 5x\,dx = 0$$

4. Compute the Fourier transform of the even function

$$f(x) = \begin{cases} 1 & -3 < x < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

What is the Fourier transform of the Fourier transform of f?
Be careful to give the value of the function for *every* real number.

The Fourier transform $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx = \hat{f}(s)$

$= \mathcal{F}[f](s)$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} 1 \cdot e^{-isx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{is} e^{-isx} \right]_{x=-3}^{3}$$

$$\underset{\text{or}}{\overset{\uparrow}{=}} \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} \underbrace{\cos sx}_{\text{even}} - i \underbrace{\sin sx}_{\text{odd}}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{3} \cos sx\, dx = \frac{2}{\sqrt{2\pi}} \frac{1}{s} \sin sx \Big|_{x=0}^{3}$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{3 \sin 3s}{3s} \quad \to 3\sqrt{\frac{2}{\pi}} \text{ as } s \to 0$$

5. Suppose that $u(r,\theta)$ is the harmonic function with continuous boundary values on the unit circle given in polar coordinates by

$$u(1,\theta) = \sin\theta + \cos 2\theta. \rightarrow u(r,\theta) = r^1 \sin 1\cdot\theta + r^2 \cos 2\theta$$

Evaluate $u\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

$$u\left(\tfrac{1}{2}, \tfrac{\pi}{2}\right) = \tfrac{1}{2}\sin\tfrac{\pi}{2} + \left(\tfrac{1}{2}\right)^2 \cos\left(2\cdot\tfrac{\pi}{2}\right)$$

Found sol^{n}s $\quad 1, \quad r^n \cos n\theta, \quad r^n \sin n\theta$

$$\rightarrow u(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right)$$

$$u(1,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) \overset{\text{want}}{=} f(\theta)$$

Fourier coeff for $f(\theta)$

6. Suppose that f is continuous on $[0, \pi]$ and that $\int_0^\pi f(x) \sin nx \, dx = 0$ for every positive integer n. Explain why f must be identically zero. b_n's $= 0$.

Take odd 2π-periodic extension. Get f^*, sine series

Parseval's $\qquad \displaystyle\sum_1^\infty b_n^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |f^*|^2 \, dx = 0$

f^* is piesewise continuous.

Conclude: $f(x) = 0$ at pts in $(0, \pi)$ where it is cont.

So $f \equiv 0$ on $(0, \pi)$. f cont on $[0, \pi]$.

So $f(0) = 0$, $f(\pi) = 0$ too.