Review 2 Final exam Friday, May 9, 7-9 pm RHPH 164

No class Friday.

4. Compute the Fourier transform of the even function

$$f(x) = \begin{cases} 1 & -3 < x < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} e^{-isx}\, dx$$

last time ✓

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{3} \cos sx\, dx$$

What is the Fourier transform of the Fourier transform of f?
Be careful to give the value of the function for *every* real number.

key

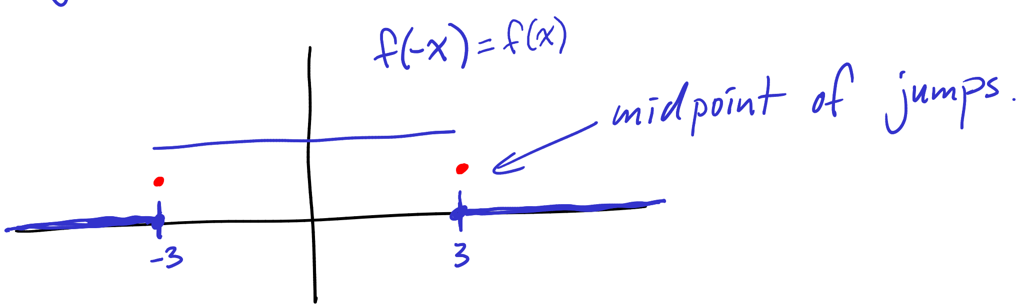
$$\mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{isx}\, ds$$

$$\mathcal{F}^{-1}[g](-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{is(-x)}\, ds = \mathcal{F}[g](x)$$

$$\mathcal{F}\big[\mathcal{F}[f]\big](x) = \mathcal{F}^{-1}\big[\mathcal{F}[f]\big](-x)$$

$$= f(-x)$$

Careful at jumps:



1. Use D'Alembert's method to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for the *infinite* vibrating string problem with initial position $u(x,0) = e^{-x^2}$ and zero initial velocity. The solution has one "hump" at time zero. Is that true of the solution for all time?

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$\frac{\partial u}{\partial t}(x,t) = c\varphi'(x+ct) - c\psi'(x-ct)$$

want

$$\begin{cases} u(x,0) = \varphi(x) + \psi(x) = e^{-x^2} \\ \frac{\partial u}{\partial t}(x,0) = c\varphi'(x) - c\psi'(x) = 0 \end{cases} \longleftarrow \varphi' \equiv \psi'$$

$$\varphi = \psi + K$$

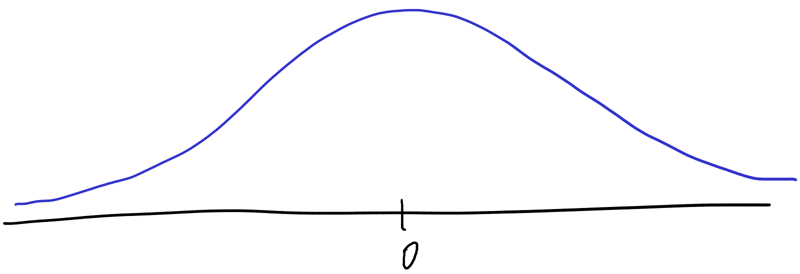
take $K=0$.

Top eqn $\varphi(x) + \varphi(x) = e^{-x^2}$

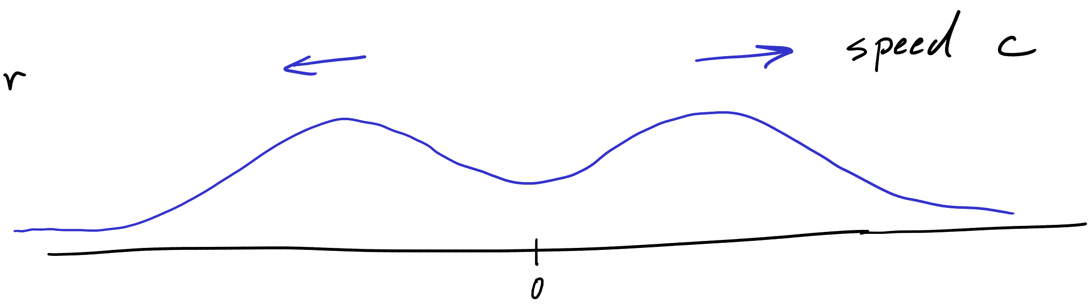
$$\varphi = \tfrac{1}{2} e^{-x^2}, \quad \psi = \tfrac{1}{2} e^{-x^2}$$

Sol^n: $$u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2} \right]$$

$t=0$



t later



2. Find the complex Fourier transform and the Fourier cosine transform of e^{-4x^2}.

$$F(s) = \tilde{\mathcal{F}}[e^{-4x^2}](s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-4x^2} e^{-isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-4x^2} \cos sx \, dx + i \cdot 0$$

is real valued

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \int_0^{\infty} e^{-4x^2} \cos sx \, dx$$

$$F'(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-4x^2} \left[-ix e^{-isx}\right] dx = i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left[-x e^{-4x^2}\right]}_{\frac{d}{dx}\left[\underbrace{\frac{1}{8} e^{-4x^2}}_{f}\right]} e^{-isx} \, dx$$

$$= i \tilde{\mathcal{F}}[f'] = i \left(is \tilde{\mathcal{F}}[f] \right)$$

$$= -\frac{1}{8} s \, F(s)$$

ODE $\quad F'(s) + \frac{1}{8} s \, F(s) = 0 \qquad u = e^{\int \frac{1}{8} s \, ds} = e^{\frac{1}{16} s^2}$ Int. Factor

$$\frac{d}{ds}\left[e^{s^2/16} F(s)\right] \equiv 0$$

$$F(s) = C e^{-s^2/16} \qquad C = F(0).$$

$$F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-4x^2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \left[\tfrac{1}{2} du\right] \qquad \begin{matrix} u = 2x \\ du = 2dx \end{matrix}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \sqrt{\pi} = \frac{1}{2\sqrt{2}}$$

$$F(s) = \frac{1}{2\sqrt{2}} e^{-s^2/16}$$

6. The Legendre polynomials

$$p_n(x) = \frac{d^n}{dx^n}(1-x^2)^n$$

are orthogonal on $[-1, 1]$. Let

$$\varphi_n = \frac{p_n}{\sqrt{A_n}} \quad \text{so that} \quad \sqrt{\int_{-1}^{1} \varphi_n^2\,dx} = 1$$

$$A_n = \int_{-1}^{1} p_n^2\,dx.$$

To expand a function f in terms of Legendre polynomials, $f = \sum_{n=0}^{\infty} c_n p_n$, what must the coefficients c_n be? What is Bessel's inequality in this context? It can be shown that the linear span of the Legendre polynomials of degree N or less is equal to the vector space of polynomials of degree N or less. Explain why this implies that Bessel's inequality must be an equality when f is assumed to be continuous.

$$f \underset{\substack{\uparrow \\ \text{hope}}}{=} \sum_{n=1}^{\infty} c_n \varphi_n \quad \text{where} \quad c_n = \langle f, \varphi_n \rangle = \int_{-1}^{1} f(x)\varphi_n(x)\,dx$$

$$\underbrace{\left\| f - \sum_{n=1}^{N} c_n\varphi_n \right\|^2}_{\| f\|^2 - \sum_1^N |c_n|^2} \leq \left\| f - \underbrace{\sum_{n=1}^{N} A_n\varphi_n}_{} \right\|^2$$

$$0 \leq \|f\|^2 - \sum_{1}^{N} |c_n|^2 \leq$$

can make this equal a Weierstraß poly close to f when f is continuous

$$\left\| f - \sum_{n=1}^{N+m} c_n\varphi_n \right\| \leq \text{same thing because can take } A_{N+1}, A_{N+2}, \ldots, A_{N+m} \text{ all zero}$$

Conclude $\displaystyle\sum_{n=1}^{\infty} c_n^2 = \|f\|^2 = \int_{-1}^{1} f(x)^2\,dx$

5. Given that $f(x) = \displaystyle\int_0^\infty \frac{\cos sx}{1+s^2}\, ds$, what is the Fourier transform of $f(x)$? Explain.

Hint: Even, odd functions. Inverse Fourier Transform.

$$f(x) = \frac{1}{2}\left[\int_{-\infty}^{\infty} \frac{\cos sx}{1+s^2}\, ds - i \underbrace{\int_{-\infty}^{\infty} \frac{\sin sx}{1+s^2}\, ds}_{0}\right] \frac{\sqrt{2\pi}}{\sqrt{2\pi}}$$

$$\frac{\sqrt{2\pi}}{2}\, \mathcal{F}\left[\frac{1}{1+s^2}\right](x)$$

Fact: $\mathcal{F}\big[\mathcal{F}[g]\big](x) = \mathcal{F}^{-1}\big[\mathcal{F}[g]\big](-x) = g(-x)$

So $\mathcal{F}[f] = \sqrt{\frac{\pi}{2}}\, \mathcal{F}\left[\mathcal{F}\left[\frac{1}{1+s^2}\right]\right](-x) = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{1+(-x)^2}$

$$= \sqrt{\frac{\pi}{2}}\, \frac{1}{1+x^2}$$