Lecture 1 MA 428 Fourier Analysis

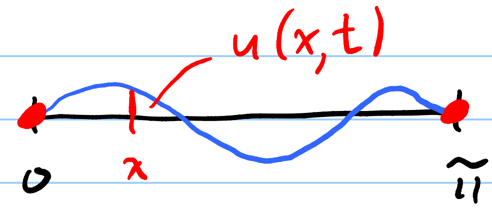
Power series: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

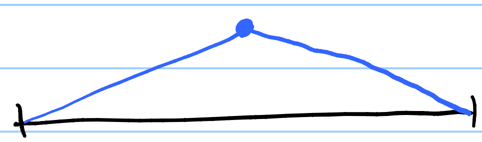
Fourier Series: $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \sin nx + b_n \cos nx)$

Bernouli v.s. Bernouli

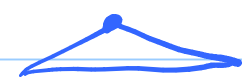
Harpsichord problem:



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ (Assume $c=1$)



Initial Condition

$u(x,0) =$  $= f(x)$

IC #2: Starts from rest: $\frac{\partial u}{\partial t}(x,0) = 0$

Jacob: $u(x,t) = \varphi(x+t) - \psi(x-t)$

Johann: Try $u(x,t) = X(x)T(t)$

Plug in: $\frac{\partial^2}{\partial t^2}\left[X(x)T(t)\right] = \frac{\partial^2}{\partial x^2}\left[X(x)T(t)\right]$

$$X(x)T''(t) = X''(x)T(t)$$

$$\boxed{\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}} = \lambda$$

a const.

(Let $x = \frac{\pi}{2}$, and t vary. Let $t = 1$ min, and x vary.)

Get 2 ODEs: $\begin{cases} X'' - \lambda X = 0 \\ T'' - \lambda T = 0 \end{cases}$

Boundary Conditions: $\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$

$$u(x,t) = X(x)T(t)$$

$$u(0,t) = X(0)\underbrace{T(t)}_{} = 0$$

want

don't want $T(t) \equiv 0$. (zero sol^n^)

$\boxed{\text{Need } X(0) = 0.}$

want

$$u(\pi,t) = X(\pi)T(t) = 0$$

$\boxed{X(\pi) = 0 \text{ too}}$

Sturm-Liouville Problem:

$$\underbrace{X'' - \lambda X = 0}_{\substack{2^{nd}\ \text{order} \\ \text{const. coeff linear ODE}}} \qquad \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases} \quad \text{B.C.'s}$$

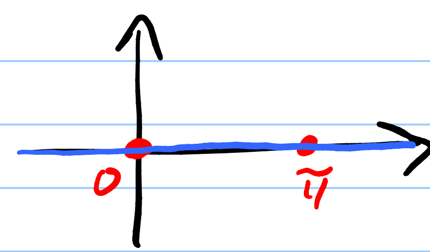
Try $X = e^{rx}$: $\quad r^2 e^{rx} - \lambda e^{rx} = 0$

$$\underbrace{(r^2 - \lambda)}_{r^2 - \lambda = 0 \quad \text{Char. Eqn.}} e^{rx} = 0$$

Case $\lambda = 0$: $X'' = 0$

$X' = c_1$

$X(x) = c_1 x + c_2$



BC's demand $X(x) \equiv 0$. Oops. Zero Sol^n.

Case $\lambda > 0$: Write $\lambda = k^2$.

$r^2 - \lambda = 0$

$r^2 - k^2 = 0$

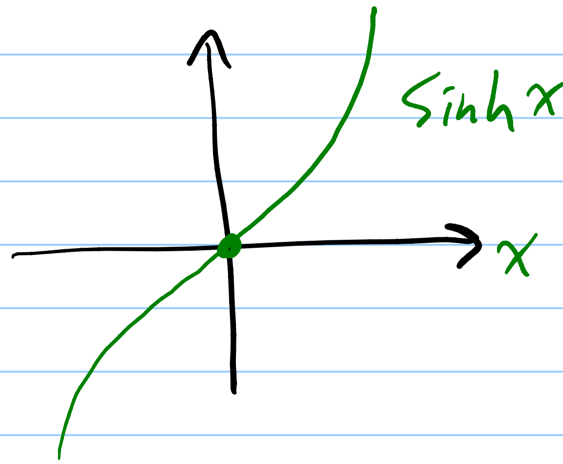
$r = \pm k$

Get $X(x) = c_1 e^{kx} + c_2 e^{-kx}$

Want $X(0) = c_1 e^0 + c_2 e^0 = 0$ ← want

$c_1 + c_2 = 0 \qquad \boxed{c_2 = -c_1}$

$$\mathbb{X}(x) = 2c_1 \left(\frac{e^{kx} - e^{-kx}}{2} \right) = 2c_1 \sinh kx$$



Ouch!

$\sinh k\pi \neq 0.$

Must have $c_1 = 0$ to make $\mathbb{X}(\pi) = 0.$

Once again, get only the zero sol^n.

Case $\lambda < 0$: Write $\lambda = -k^2$.

$r^2 - \lambda = 0$

$r^2 + k^2 = 0$

$\boxed{r = \pm ki}$

$$e^{(ki)x} = \cos kx + i \sin kx$$

complex sol^n

two real sol^{n}s

Gen^l Sol^n: $\mathbb{X}(x) = c_1 \cos kx + c_2 \sin kx$

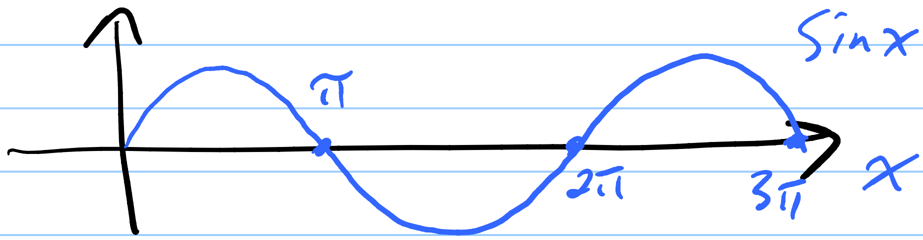
Need $\mathbb{X}(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$ want

So $\mathbb{X}(x) = c_2 \sin kx.$

Finally, need $\mathbb{X}(\pi) = c_2 \sin k\pi = 0$ want

Note: Don't want $c_2 = 0$. (Get zero sol^n).

Need $\boxed{\sin K\pi = 0}$



zeroes at $n\pi$, $n = 1, 2, 3, \ldots$

Must have $K\pi = n\pi$, $n = 1, 2, 3, \ldots$

Get: K must be $K = 1, 2, 3, 4, \ldots$

So $\lambda = -K^2 = -n^2$ $\quad n = 1, 2, 3, \ldots$

Back to T problem: $T'' - \lambda T = 0$

$\lambda = -n^2$

$$T(t) = A_n \cos nt + B_n \sin nt$$

For $\lambda = -n^2$: $\quad u_n(x,t) = X_n(x)\, T_n(t)$

$$= c \sin nx \left(A_n \cos nt + B_n \sin nt \right)$$

take $c = 1$.

Big idea: Add up sol^n's to get more.

Linear PDE. BC's are homogeneous.

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos nt + B_n \sin nt\right) \sin nx$$

Finally: Initial Cond:

want

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = f(x)$$

want

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n B_n \sin nx = 0$$

Take all B_n's $= 0$.

Sol^n: $$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos nt}$$