Lecture 10 Fejer's Theorem

Last time: If f is 2π periodic and Lipschitz smooth (or C^1-smooth), then $S_N(x) \to f(x)$ at each x.

This time: If f 2π periodic and C^2-smooth, then $S_N \to f$ uniformly and absolutely.

Stein notation:

$$\hat{f}(n) = c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

Hmmm. $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$

$$f'(x) \overset{?}{=} \sum_{-\infty}^{\infty} in\,c_n e^{inx}$$

$$f''(x) \overset{?}{=} \sum_{-\infty}^{\infty} \underbrace{(in)^2}_{-n^2} c_n e^{inx}$$

Is $\widehat{f'}(n) \stackrel{?}{=} in\,\hat{f}(n)$ Yes!

$\widehat{f''}(n) \stackrel{?}{=} -n^2 \hat{f}(n)$

Why:
$$\widehat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{e^{-inx}}_{u}\ \underbrace{f'(x)\,dx}_{dv}$$

$du = -ine^{-inx}$ $\quad v = f(x)$

$$= \frac{1}{2\pi}\Big[\underbrace{uv\Big|_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} v\,du\Big]$$

because u, v 2π periodic

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[-ine^{-inx}\right]dx$$

$$= in\,\hat{f}(n) \checkmark$$

or
$$\widehat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\,\underbrace{(\cos nx - i\sin nx)}_{e^{-inx}}\,dx$$

and do real integrals.

Similarly, show $\widehat{f''}(n) = -n^2 \hat{f}(n)$.

Aha! $\left|\widehat{f''}(n)\right| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f''(x)\, e^{-inx}\, dx\right|$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|f''|}_{\leq M}\ \underbrace{|e^{-inx}|}_{=1}\, dx$$

f C^2-smooth

$$\leq \frac{1}{2\pi}\cdot M \cdot 1 \cdot 2\pi \leq M$$

Big consequence: f C^2-smooth, then

$$\left|\hat{f}(n)\right| \leq \frac{M}{n^2}$$

Integral test shows $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

<u>Thm</u>: Compare tail end of F.S. to $\sum \frac{M}{n^2}$ to see $S_N \to f$ unif and absolutely!

$$\left|c_n e^{inx}\right| = \underbrace{|c_n|}_{M/n^2}\ \underbrace{|e^{inx}|}_{1}$$

Cesáro means:

$$\sigma_N(x) = \frac{s_0(x) + s_1(x) + \cdots + s_{N-1}(x)}{N}$$

Fejer's Thm: If f 2π periodic and continuous, then $\sigma_N \to f$ uniformly.

Fejer kernel:

$$S_n(x) = \int_{-\pi}^{\pi} D_N(t)\, f(x-t)\, dt$$

$$\sigma_N(x) = \int_{-\pi}^{\pi} \underbrace{\frac{D_0(t) + D_1(t) + \cdots + D_{N-1}(t)}{N}}_{\text{Fejér kernel } F_N(t)} f(x-t)\, dt$$

Very good kernel.

$$F_N(x) = \frac{1}{2\pi N} \frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$$

Properties:

1) $F_N(x) \geq 0$ ✓

2) L'H: $F_N(0) = \frac{1}{2\pi N}\left(\frac{N/2}{1/2}\right)^2 = \frac{N}{2\pi}$ ✓

3) $\int_{-\pi}^{\pi} F_N(x)\,dx = 1$ ✓

because $\int_{-\pi}^{\pi} D_N\,dx = 1$.

4) Let $I_\delta = [-\pi,-\delta] \cup [\delta,\pi]$.

$F_N(x) \to 0$ uniformly on I_δ.

$$F_N(x) = \underset{\uparrow\,!}{\frac{1}{2\pi N}} \underbrace{\frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}}_{\text{bounded on } I_\delta}$$

Bad thing about D_N: $\int_{-\pi}^{\pi} |D_N|\,dx \sim \log N$

Proof of Fejér's:

$$\sigma_N(x) - f(x) = \int_{-\pi}^{\pi} F_N(t)\left[f(x+t) - f(x)\right]dt$$

$$= \int_{I_\delta} \underbrace{F_N(t)}_{\text{small}}\left[f(x+t) - f(x)\right]dt \qquad I_1$$

$$+ \int_{-\delta}^{\delta} F_N(t) \Big[\underbrace{f(x+t) - f(x)}_{\text{small}} \Big] dt \quad I_2$$

Let $\varepsilon > 0$. f cont on $[-\pi, \pi]$. So $|f|$ assumes its max M. And f is uniformly continuous. So $\exists\ \delta$ such that

$$|f(x+t) - f(x)| < \frac{\varepsilon}{2} \text{ if } |t| < \delta.$$

Now $$|I_2| = \left| \int_{-\delta}^{\delta} \right| \leq \int_{-\delta}^{\delta} \underset{\geq 0!}{F_N(t)} \underbrace{|f(x+t) - f(x)|}_{\leq \frac{\varepsilon}{2}} dt$$

$$\leq \frac{\varepsilon}{2} \int_{-\delta}^{\delta} F_N(t)\, dt \leq \frac{\varepsilon}{2} \underbrace{\int_{-\pi}^{\pi} F_N(t)\, dt}_{= 1}$$

$$\leq \frac{\varepsilon}{2}$$

Next $$|I_1| = \left| \int_{I_\delta} F_N(t) \big[f(x+t) - f(x) \big] dt \right|$$

$$\leq \int_{I_\delta} \underset{\geq 0!}{F_N(t)} \Big[\underbrace{|f(x+t)| + |f(x)|}_{\leq 2M} \Big] dt$$

$$\leq \left(\max_{I_\delta} F_N \right) \cdot 2M \cdot (2\pi - 2\delta)$$

$$\longrightarrow 0 \quad \text{as } N \to \infty.$$

Make it $< \frac{\varepsilon}{2}$ for $N > N_0$.

Thm: Weierstraß Thm: Polynomials are dense in continuous fcns on $[a,b]$.

[f cont. Given $\varepsilon > 0$, $\exists$ poly p such that $|f - p| < \varepsilon$ on $[a,b]$.]

Fejér's Thm: Trigonometric polynomials are dense among 2π periodic continuous fcns.

Consequence: Bessel's Ineq becomes the Parseval's Identity for F.S.