Lecture 11 Applications of Fejér's Thm.

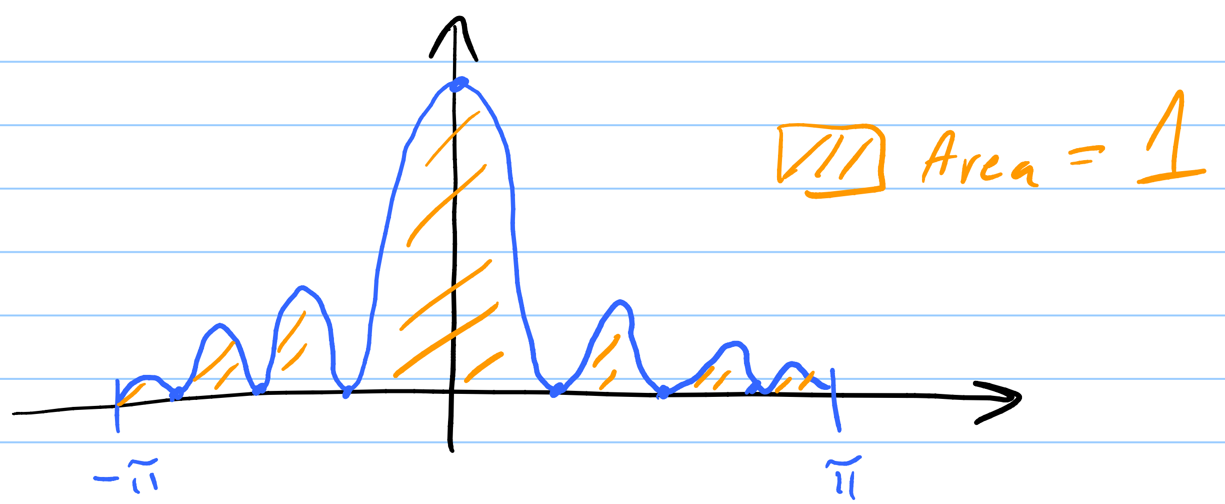
$$S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

$$= \sum_{n=-N}^{N} c_n e^{inx}$$

Cesáro means:

$$\sigma_N(x) = \frac{S_0(x) + \cdots + S_{N-1}(x)}{N}$$

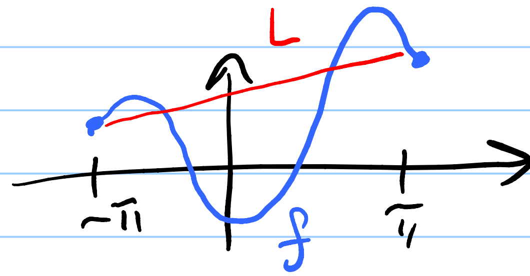
$$= \int_{-\pi}^{\pi} \underbrace{F_N(t)}_{\text{Fejér kernel}} f(x-t)\, dt$$



<u>Fejér's Thm</u>: $\sigma_N \to f$ uniformly if f is 2π periodic and continuous.

Fun fact: Fejér's $\Rightarrow$ Weierstraß.

Why: Suppose f continuous on $[a,b]$. Make a linear change of vars so that f is cont. on $[-\pi,\pi]$.



Replace f by $f-L$ so that $f(-\pi)=f(\pi)=0$.

Extend f as a cont. 2π-periodic fcn on $\mathbb{R}$.

Let $\varepsilon>0$. Fejér's $\Rightarrow$ $\exists$ Trig poly P_N such that $|f-P_N|<\frac{\varepsilon}{2}$ on $[-\pi,\pi]$.

Aha! $\quad \cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots$ unif conv on $[-\pi,\pi]$

$\quad \sin\theta = \theta-\frac{\theta^3}{3!}+\cdots$ " " "

Get poly approx for terms in P_N. Get a poly p with $|P_N-p|<\frac{\varepsilon}{2}$.

$$|f-p| = |f-P_N+P_N-p|$$
$$\leq |f-P_N|+|P_N-p| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \checkmark$$

Bessel's Ineq is an equality for F.S.

φ_n's $\quad \langle \varphi_n, \varphi_m \rangle = 0$ if $n \neq m$

$$\|\varphi_n\|^2 = \langle \varphi_n, \varphi_n \rangle = 1 \leftarrow \text{orthonormal}$$

Hmmm: $\quad \langle e^{int}, e^{int} \rangle = \int_{-\pi}^{\pi} \underbrace{e^{int}\,\overline{e^{int}}}_{\underbrace{e^{int}e^{-int}}_{1}}\,dt = 2\pi$

Fact: $\frac{1}{\sqrt{2\pi}} e^{inx}$

are orthonormal.

Bessel's: $\quad f \sim \sum_{1}^{\infty} c_n \varphi_n \qquad c_n = \langle f, \varphi_n \rangle$

$$E_N^* = \left\| f - \sum_{1}^{N} c_n \varphi_n \right\|^2 =$$

$$\int \left| f - \sum_{1}^{N} c_n \varphi_n \right|^2 dx = \|f\|^2 - \sum |c_n|^2 \geq 0$$

$$E_N = \left\| f - \sum_{n=1}^{N} A_n \varphi_n \right\|^2$$

$A_n \uparrow$ not Fourier coeff

Fourier Series approx best in L^2. $\quad E^* \leq E$

with equality exactly when $\forall$ $c_n = A_n$.

Aha! Fejér's $\Rightarrow$ $\exists$ Trig poly

P_{N_0} such that $|f - P_{N_0}| < \varepsilon$. Then

$$\|f - P_{N_0}\|^2 = \int_{-\pi}^{\pi} \underbrace{|f - P_{N_0}|^2}_{<\varepsilon^2} dx < \varepsilon^2 \cdot 2\pi$$

Now $E_N^* \leq E_{N_0} < \varepsilon^2 2\pi$ for $N \geq N_0$

Conclude $E_N^* \to 0$ as $N \to \infty$.

$$\|f\|^2 - \sum_1^\infty |c_n|^2 = 0$$

Parseval's Identity.

<u>Think</u>: $f = \sum_1^\infty c_n \varphi_n$

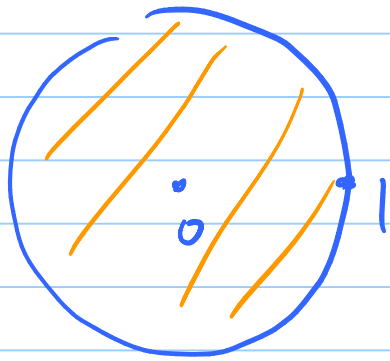
coordinates!

$$\|f\| = \sqrt{\int_{-\pi}^{\pi} |f|^2 dx} = \underbrace{\sqrt{\sum_1^\infty |c_n|^2}}_{\text{length!}}$$

$$(f, g) = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx = \sum_{1}^{\infty} \underset{\substack{\uparrow \\ f \\ \text{coord}}}{c_n}\ \underset{\substack{\uparrow \\ g \\ \text{coord}}}{\overline{b_n}}$$

Just like $\mathbb{R}^n$

Steady State heat problem on a disc



$$\underbrace{\frac{\partial u}{\partial t}}_{\substack{\to 0 \\ \text{at steady} \\ \text{state}}} = c\,\Delta u$$

$$\begin{cases} \Delta u \equiv 0 & \text{PDE} \\ u(1,\theta) = f(\theta) \leftarrow \text{given} & \text{Boundary Cond.} \end{cases}$$

Dirichlet Problem for disc.

Look for sol'ns $u(r,\theta) = R(r)\,\Theta(\theta)$

Polar form $\underbrace{\Delta u}_{u_{xx}+u_{yy}} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

PDE $\quad R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$

want

$$r^2 R''\Theta + rR'\Theta = -R\Theta''$$

Divide by R and Θ

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Hmmm: $\quad u(r,\theta) = R(r)\Theta(\theta)$

Hidden B.C.: Need $\Theta(-\pi) = \Theta(\pi)$

$\Theta'(-\pi) = \Theta'(\pi)$

Θ prob: $\quad \Theta'' + \lambda\Theta = 0$

Θ periodic

$\lambda < 0$. Hyperbolic Sin, Cos. No way.

$\lambda = 0$. $\quad \Theta(\theta) = A\theta + B$ $\quad$ Get Constant Sol^n B

need $A = 0$

$\lambda > 0$: Need $\sin n\theta$, $\cos n\theta$

Get non-zero periodic sol^{n}s exactly when $\lambda = n^2$, $n = 1, 2, \cdots$

Get $\Theta_0(\theta) = a_0$

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

Find $R_n(r)$'s:

Case $n=0$: $\dfrac{r^2 R'' + r R'}{R} = 0$

$$r^2 R'' + r R' = 0 \qquad \text{Let } v = R'.$$

$$R'' + \frac{1}{r} R' = 0$$

$$v' + \frac{1}{r} v = 0$$

First order linear!

Int factor $e^{\int \frac{1}{r} dr} = e^{\ln r} = r$

$$\underbrace{r\left[v' + \frac{1}{r} v\right]}_{\frac{d}{dr}[rv]} = 0$$

$$\frac{d}{dr}[rv] \equiv 0$$

So $\quad rv = c_1$

$$v = \frac{c_1}{r}$$

So $\quad R' = \frac{c_1}{r}$ $\qquad$ ouch! $c_1 = 0.$

$$R_0(r) = c_1 \ln r + c_2$$

$R_0(r)$ is constant. Take $R_0(r) \equiv 1$.

So $\quad u_0(r,\theta) = R_0(r)\,\Theta_0(r) = 1 \cdot a_0 = a_0.$