Lecture 12 Dirichlet problem on disc

$$u_n(r,\theta) = \underbrace{R_n(r)}_{\uparrow}\ \underbrace{\Theta_n(\theta)}_{a_n \cos n\theta + b_n \sin nx}$$

$$\frac{r^2 R'' + rR'}{R} = \lambda = n^2$$

$$r^2 R'' + rR' - n^2 R = 0$$

ODEs: 1) $ay'' + by' + cy = 0$ Const coeff linear homog 2^{nd} order ODE

Try $y = e^{\mu x}$

2) $\alpha x^2 y'' + \beta x y' + cy = 0$ Euler equation

Try $y = x^p$

$y' = p x^{p-1}$

$y'' = p(p-1)x^{p-2}$

Plug in and force it!

want

$$\alpha x^2 p(p-1)x^{p-2} + \beta x p x^{p-1} + cx^p = 0$$

$$\Big[\underbrace{\alpha p(p-1)+\beta p+c}\Big]x^p=0$$

Char. Poly. Need it $=0$.

Case 1: p_1, p_2 real and $\neq$.

Gen^l Sol^n: $y=c_1x^{p_1}+c_2x^{p_2}$

Case 2: Repeated, only one real root p.

Get $y_1=x^p$. Reduction of order:

Try $y_2=ux^p$. See $u=\ln x$.

Gen^l Sol^n: $y=c_1x^p+c_2x^p\ln x$

Case 3: $p=a\pm bi$

Complex sol^n $x^{a+bi}=x^a x^{bi}$

$$=x^a e^{bi\ln x}$$

$$=x^a\left(\cos b\ln x+i\sin b\ln x\right)$$

$$= \underbrace{x^a \cos(b \ln x)} + i \underbrace{x^a \sin(b \ln x)}$$

two real sol^{n}s!

Gen^l Sol^n $\quad y = c_1 y_1 + c_2 y_2$

Back to $\quad r^2 R'' + r R' - n^2 R = 0.$

Try $R(r) = r^p$:

$$p(p-1) + p - n^2 = 0$$

$$p^2 - n^2 = 0$$

$$p = \pm n$$

Gen^l Sol^n $\quad R(r) = c_1 r^n + c_2 r^{-n}$

ouch! c_2 must $= 0$

Get $\quad R_n(r) = r^n.$

Now $\quad u_0(r,\theta) = a_0$

$$u_n(r,\theta) = r^n \left(a_n \cos n\theta + b_n \sin n\theta \right)$$

Let $\quad u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta \right)$

Last thing: Boundary Condition

$$u(1,\theta) = \underbrace{a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right)}_{\text{Fourier Series for } f\,!} \overset{\text{want}}{=} f(\theta)$$

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \right) r^{|n|} e^{in\theta}$$

$$= \int_{-\pi}^{\pi} \underbrace{\left(\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)} \right)}_{P(re^{i\theta},\, e^{it})} f(t)\, dt$$

Poisson Kernel

Write $p(r,\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty} r^n e^{in\theta} + \frac{1}{2\pi}\sum_{n=1}^{\infty} r^n e^{-in\theta}$$

Aha! $\quad 1+z+z^2+\cdots+z^N = \dfrac{1-z^{N+1}}{1-z} \qquad z = re^{\pm i\theta}$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{if } |z|<1.$$

$$p(r,\theta) = \frac{1}{2\pi}\frac{1}{1-re^{i\theta}} + \frac{1}{2\pi} re^{-i\theta}\left(1+(re^{-i\theta})+\cdots\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}}\right]$$

$$= \frac{(1-re^{-i\theta}) + re^{-i\theta} - r^2 \underbrace{e^{i\theta}e^{-i\theta}}_{1}}{2\pi \underbrace{(1-re^{i\theta})(1-re^{-i\theta})}_{|1-re^{i\theta}|^2}} \qquad w\bar{w} = |w|^2$$

$$= \frac{1-r^2}{2\pi\,|1-re^{i\theta}|^2}$$

$$u(r,\theta) = \int_{-\pi}^{\pi} \frac{1}{2\pi}\,\frac{1-r^2}{|1-re^{i(\theta-t)}|^2}\, f(t)\,dt$$

Poisson Integral Formula

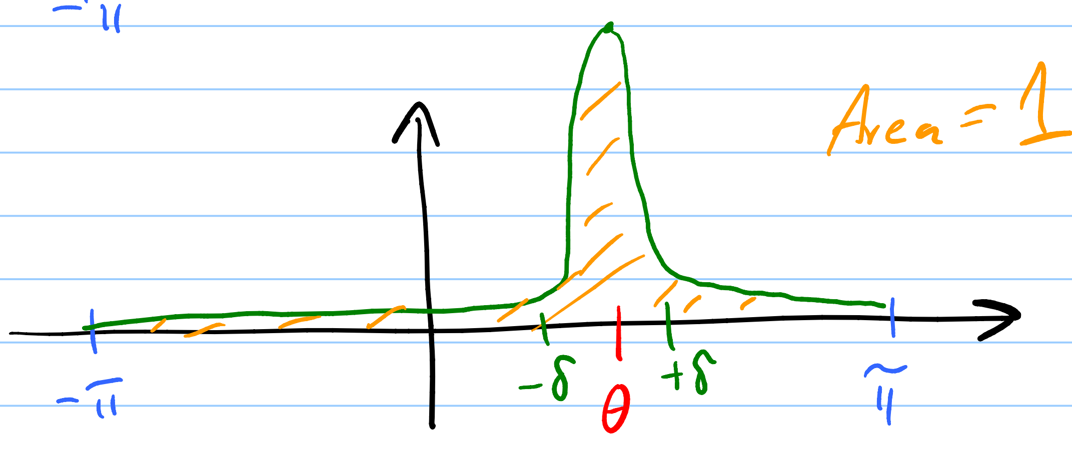
$$= \int_{-\pi}^{\pi} \underbrace{\frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}}_{\text{Poisson kernel} \quad P(re^{i\theta}, e^{it})} f(t)\,dt$$

Good Kernel!

1) $P > 0$

2) $\int_{-\pi}^{\pi} P(re^{i\theta}, e^{it})\,dt = 1$

3)



$P(re^{i\theta}, e^{it}) \to 0$ uniformly on

$I_\delta = [-\pi, \pi] - (\theta-\delta, \theta+\delta)$ as $r \to 1$.

$P(re^{i\theta}, e^{it})$ is harmonic in r, θ.