Lecture 13 Poisson Formula for D. Prob.

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$ "converges" means

$$\sum_{n=1}^{\infty} c_{-n} e^{-in\theta} \text{ and } \sum_{n=0}^{\infty} c_n e^{in\theta}$$ both

converge.

$$a_0 + \sum_{n=1}^{N} \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

$$= \sum_{n=-N}^{N} c_n e^{in\theta}$$

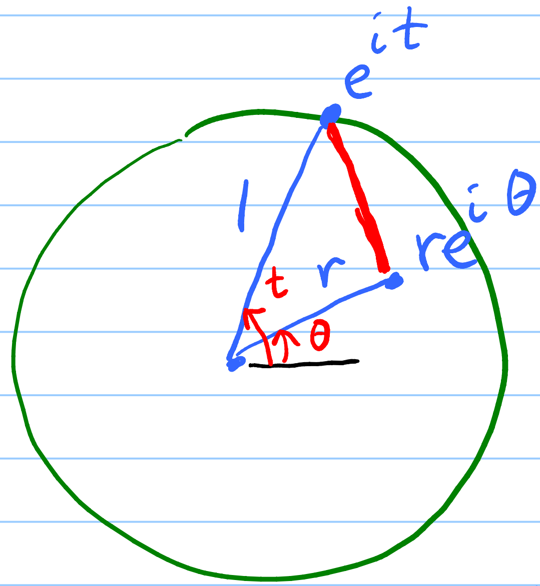
$$u(r,\theta) = \lim_{N \to \infty} \left[a_0 + \sum_{n=1}^{N} \left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right) \right]$$

$$= \lim_{N \to \infty} \left[\sum_{n=-N}^{N} c_n r^{|n|} e^{in\theta} \right]$$

$$= \int_{-\pi}^{\pi} P(r,\theta,t) f(t)\, dt$$

$$P(r,\theta,t) = \frac{1-r^2}{2\pi\left|1-re^{i(\theta-t)}\right|^2}$$ ← Poisson kernel

$$\underbrace{\left|e^{it}\right|^2}_{1}\cdot\left|1-re^{i(\theta-t)}\right|^2 = \underbrace{\left|e^{it}-re^{i\theta}\right|^2}_{\substack{\text{law of cosines} \\ = \\ 1^2-2r\cdot 1\cdot\cos(\theta-t)+r^2}}$$



$$P(r,\theta,t) = \frac{1-r^2}{2\pi\left(1-2r\cos(\theta-t)+r^2\right)}$$

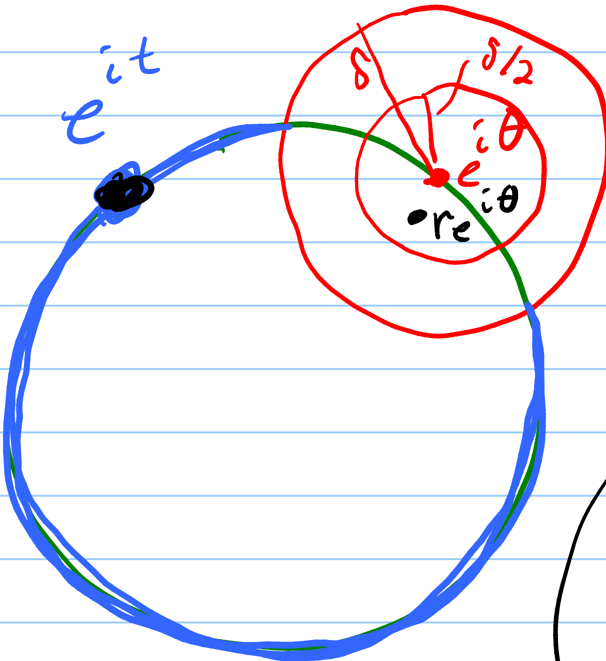
<u>Key properties:</u> 1) $P>0$. $(r<1)$. ✓

2) $\displaystyle\int_{-\pi}^{\pi} P(r,\theta,t)\,dt = \underline{1}$ $\quad(r<1)$

Why: $P(r,\theta,t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|}\underbrace{e^{in(\theta-t)}}_{\int=0 \text{ if } n\neq 0}$

$$\int P\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{r^0}_{1}\,dt = \frac{1}{2\pi}\cdot 2\pi = 1.$$

3)



$$P(r,\theta,t) = \frac{1}{2\pi}\,\frac{1-r^2}{|e^{it}-re^{i\theta}|^2}$$

$$|e^{it} - re^{i\theta}| > \frac{\delta}{2}$$

$$\le \frac{1}{2\pi}\,\frac{1-r^2}{\left(\frac{\delta}{2}\right)^2}$$

$$\le \frac{1}{2\pi}\,\frac{(1-r)\overbrace{(1+r)}^{\le 2}}{\left(\frac{\delta}{2}\right)^2} \le \frac{4}{\pi\delta^2}(1-r) \quad \text{on blue part}$$

$$I_\delta = \{t : e^{it} \notin D_\delta(e^{i\theta})\}.$$

4) $\Delta_{r,\theta} P(r,\theta,t) \equiv 0$ ✓

Thm: $u(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta,t) f(t)\,dt$

solves the D. Prob:

$\Delta u \equiv 0$ on $D_1(0)$

u cont on $\overline{D_1(0)}$ and

$u(1,\theta) = f(\theta) \leftarrow$ cont 2π periodic given fcn.

Pf: $\Delta u \equiv 0$ because ok to diff under $\int$ and property 4.

$$u(r,\theta) - f(\theta) =$$

$$\int_{-\pi}^{\pi} P(r,\theta,t) f(t)\,dt - f(\theta) \underbrace{\int_{-\pi}^{\pi} P(r,\theta,t)\,dt}_{1}$$

$$= \int_{-\pi}^{\pi} P(r,\theta,t)\left[f(t) - f(\theta)\right] dt$$

$$= \int_{I_\delta} \underbrace{P(r,\theta,t)}_{\text{small}} \underbrace{\left[f(t) - f(\theta)\right]}_{\text{bounded by } 2M} dt +$$

$$\int_{-\delta}^{\delta} P(r,\theta,t) \underbrace{\left[f(t) - f(\theta) \right]}_{< \frac{\varepsilon}{2} \text{ by continuity}} dt$$

$$= I_1 + I_2 .$$

$$|I_2| \le \int_{-\delta}^{\delta} \overset{>0!}{P(r,\theta,t)} \underbrace{|f(t) - f(\theta)|}_{<\varepsilon/2} dt$$

$$\le \frac{\varepsilon}{2} \int_{-\delta}^{\delta} P(r,\theta,t)\, dt$$

$$\le \frac{\varepsilon}{2} \underbrace{\int_{-\pi}^{\pi} P(r,\theta,t)\, dt}_{1} \le \frac{\varepsilon}{2} .$$

Finally, make $|I_1| < \frac{\varepsilon}{2}$ by making $re^{i\theta} \longrightarrow e^{i\theta}$ by letting $r \to 1$ inside $D_\delta(e^{i\theta})$.

<u>HWK 4</u> : Chap 1 : 8. hints.

F is C^2-smooth. Fix x.

$$DQ = \frac{F'(x+h) - F'(x)}{h} = F''(x) + \psi(h) \quad (*)$$

where $\psi(h) \to 0$ as $h \to 0$.

Define $\psi(0) = 0$.

Solve $(*)$ for $F'(x+h)$:

$$F'(x+h) = F'(x) + F''(x)h + \psi(h) \cdot h \quad (\dagger)$$

true at $h=0$ too.

Integrate $(\dagger)$ w.r.t. h. $\int_0^h$

Note $F'(x+h) = \frac{d}{dh} F(x+h)$

$$\int_0^h F'(x+\tilde{\tau})\,d\tilde{\tau} = F'(x)h + F''(x)\int_0^h \tilde{\tau}\,d\tilde{\tau} + \int_0^h \psi(\tilde{\tau})\cdot\tilde{\tau}\,d\tilde{\tau}$$

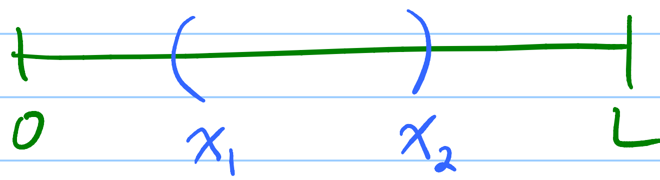
$$F(x+h) - F(x) = F'(x)\,h + F''(x)\,\frac{h^2}{2} + E(h)$$

$$F(x+h) = F(x) + F'(x)h + F''(x)\frac{h^2}{2} + E(h)$$

where $E(h)$ is <u>really</u> small.

Same formula with $-h$ in place of h.

<u>Heat problems</u>: Hot wire prob.



$u(x,t)$ = temp at time t

Heat in (x_1, x_2): $\displaystyle\int_{x_1}^{x_2} u(x,t)\,dx$

Changing at rate: $\displaystyle\frac{\partial}{\partial t}\int_{x_1}^{x_2} u(x,t)\,dx$

$$= \int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx$$

Kelvin's Law: Rate of heat flow is proportional to the temperature gradient.