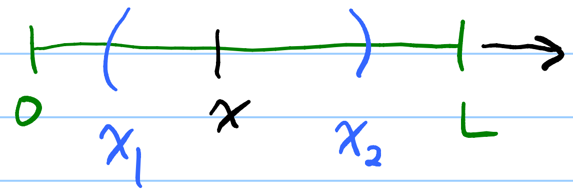
Lecture 14 Hot wire



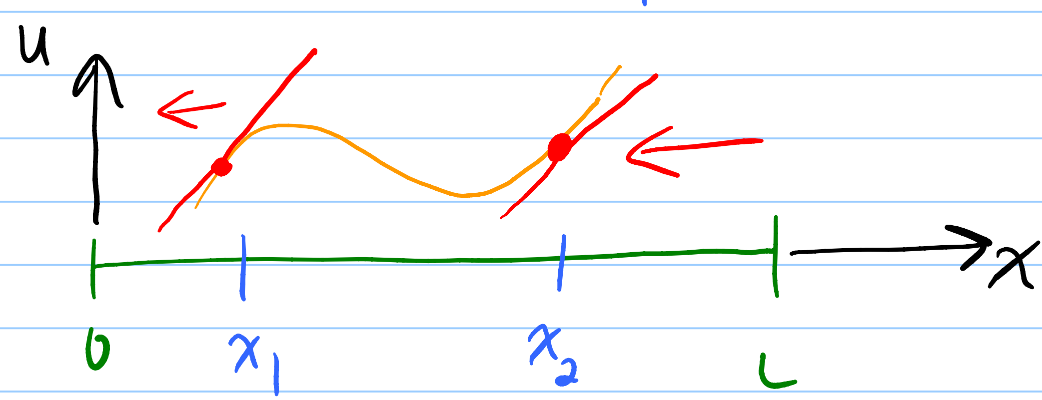
$u(x,t) =$ temp at x at time t

Heat content from x_1 to x_2:

$$c \int_{x_1}^{x_2} u(x,t)\, dx$$

Changing at rate $\frac{\partial}{\partial t}\left(c \int_{x_1}^{x_2} u(x,t)\, dx \right)$

$$= c \int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\, dx$$



Kelvin's Law: Rate of heat flow is proportional to the temp gradient.

Equate:

$$c\int_{x_1}^{x_2} \frac{\partial u}{\partial t}(x,t)\,dx = \underbrace{k\frac{\partial u}{\partial x}(x_2,t) - k\frac{\partial u}{\partial x}(x_1,t)}_{\text{Fund. Thm. Calc.}}$$

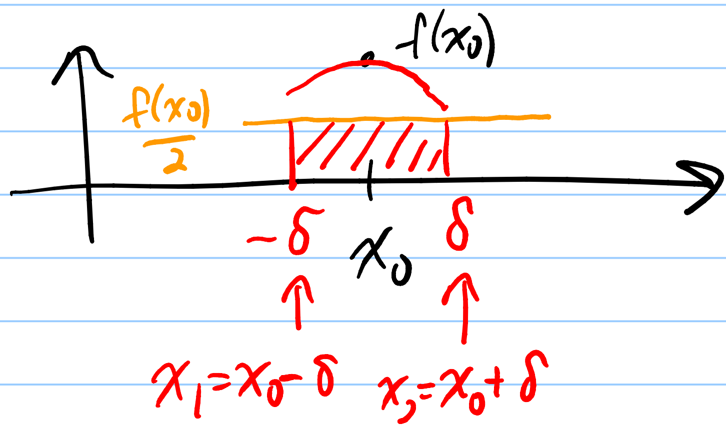
$$= k\int_{x_1}^{x_2} \frac{\partial^2 u}{\partial x^2}(x,t)\,dx$$

$$\int_{x_1}^{x_2}\left[\frac{\partial u}{\partial t} - \frac{k}{c}\frac{\partial^2 u}{\partial x^2}\right]dx = 0$$

Aha! Since x_1, x_2 arbitrary, we must conclude $[\quad] \equiv 0$.

<u>Fact</u>: If $f(x)$ is continuous on $[a,b]$ and $\int_{x_1}^{x_2} f(x)\,dx = 0$ for $\forall\, x_1, x_2$ with $a \le x_1 < x_2 \le b$, then $f \equiv 0$.

Pf: Suppose $f(x_0) \neq 0$. We may assume that $f(x_0) > 0$. [Otherwise replace f by $-f$.]

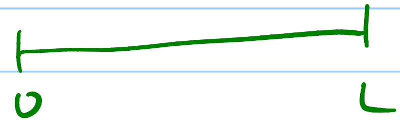


$\varepsilon = \dfrac{f(x_0)}{2}$.

$\exists\ \delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$

Fact: If f is continuous and $\int_a^b |f|^2 dx = 0$, then $f \equiv 0$.

Wire problem:



$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: $u(0,t) = 0$
$u(L,t) = 0$

Initial Condition: $u(x,0) = f(x) \leftarrow$ given

Try $u(x,t) = X(x)\,T(t)$

$$\frac{\partial u}{\partial t} = X T' \overset{\text{want}}{=} c^2 X'' T = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda, \text{ a const.}$$

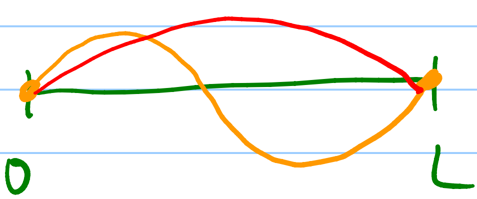
B.C. yield Sturm-Liouville prob

$$X'' - \lambda X = 0$$

$$\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$$

No non-zero sol^{n}s for case $\lambda = 0$ or $\lambda > 0$.

$\lambda < 0$:



$\lambda = -\left(\frac{n\pi}{L}\right)^2 \qquad X_n(x) = \sin\frac{n\pi x}{L}$

T prob: $\quad \frac{1}{c^2}\frac{T'}{T} = \lambda = -\left(\frac{n\pi}{L}\right)^2$

$$T' = -\left(\frac{cn\pi}{L}\right)^2 T$$

EXPONENTIAL DECAY!

Sol^n: $T_n = K e^{-\left(\frac{cn\pi}{L}\right)^2 t}$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}}$$

Finally, we want

want

I.C.: $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$

Fourier Sine Series!

Need $\boxed{A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx}$

Do it for $f(x) \equiv 1$. $c=1$. $L=\pi$

$$A_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\Big[1 - \underbrace{\cos n\pi}_{(-1)^n}\Big] = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$u(x,t) = \frac{4}{\pi}\left(e^{-1^2 t}\sin x + \frac{1}{3}e^{-3^2 t}\sin 3x + \frac{1}{5}e^{-5^2 t}\sin 5x + \cdots\right)$$