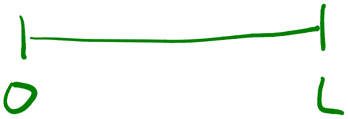
Lecture 15



Last time: refridgerated ends:

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

Got $u(x,t) = \sum_{n=1}^{\infty} A_n \underbrace{e^{-\left(\frac{cn\pi}{L}\right)^2 t}}_{\text{die away fast!}} \sin\frac{n\pi x}{L}$

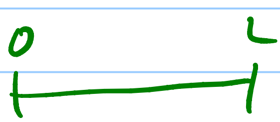
$$e^{-\left(\frac{cn\pi}{L}\right)^2 t} = \Big[\underbrace{e^{-\left(\frac{c\pi}{L}\right)t}}_{<1 \text{ if } t>0.}\Big]^{n^2}$$

$u(x,t) \to 0$ as $t \to \infty$. In fact, quite fast.

Today: New BC: Insulated ends.

Kelvin's Law: (Heat flow) prop. to ∇u

Conclude: Temp gradient $= 0$ at an insulated end.



$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{BC} \begin{cases} \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = 0 \end{cases} \quad \text{Ins. Ends.}$$

$$\text{I.C.}: \ u(x,0) = f(x) \leftarrow \text{known.}$$

Separation of variables : $u(x,t) = X(x)T(t)$

$$\boxed{\begin{matrix} X'' - \lambda X = 0 \\ X'(0) = 0, \ X'(L) = 0 \end{matrix}} \leftarrow \text{Sturm-Liouville problem.} \qquad T' - c^2\lambda T = 0$$

BC? $\quad \frac{\partial u}{\partial x} = X'(x)T(t)$

$$\frac{\partial u}{\partial x}(0,t) = \underbrace{X'(0)}_{\text{need } = 0} T(t) \overset{\text{want}}{=} 0$$

Similarly, need $X'(L) = 0$.

Case $\lambda = 0$: $\quad X'' = 0$

$$X(x) = Ax + B$$

$$X'(x) = A \leftarrow \text{Need } A = 0 \text{ to get } X'(0) = 0, \ X'(L) = 0.$$

Aha! Get a non-zero solution $X(x) \equiv B$.

Take $B = 1$.

T-problem: $T' = 0$. Get $T(t) = \text{const.}$ too.

$$\boxed{u_0(x,t) = 1}$$

Case $\lambda > 0$: $\lambda = k^2$

$$X'' - k^2 X = 0$$

$$\begin{cases} r^2 - k^2 = 0 \\ r = \pm k \end{cases}$$

$$X(x) = Ae^{-kx} + Be^{kx}$$

$$= a\cosh kx + \cancel{b\sinh kx}$$

$$X'(x) = ak\sinh kx + \cancel{bk\cosh kx}$$

$$X'(0) = 0 + bk\cdot 1 \overset{\text{want}}{=} 0$$

Need $b = 0$

$$X'(L) = ak\underbrace{\sinh kL}_{\text{not zero.}}$$

Oops. Need $a = 0$ too.

Only get the zero sol^n.

Case $\lambda < 0$: $\lambda = -K^2$

$$\begin{cases} r^2 + K^2 = 0 \\ r = \pm K i \end{cases}$$

$$X(x) = A \cos Kx + \cancel{B \sin Kx}$$

$$X'(x) = -AK \sin Kx + \cancel{BK \cos Kx}$$

$$X'(0) = 0 + BK \cdot 1 \overset{\text{want}}{=} 0$$

Need $B = 0$.

$$X'(L) = -AK \sin KL \overset{\text{want}}{=} 0$$

Don't want $A = 0$ (zero sol^n).

Aha! Need $\boxed{\sin KL = 0}$

Need $KL = n\pi \quad n = 1, 2, 3, \ldots$

$$K = \frac{n\pi}{L} \qquad \lambda = -\left(\frac{n\pi}{L}\right)^2$$

$$\boxed{X_n(x) = \cos\frac{n\pi x}{L}}$$

$$T' - \left(-c^2\left(\frac{n\pi}{L}\right)^2\right)T = 0$$

$$T_n(t) = e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

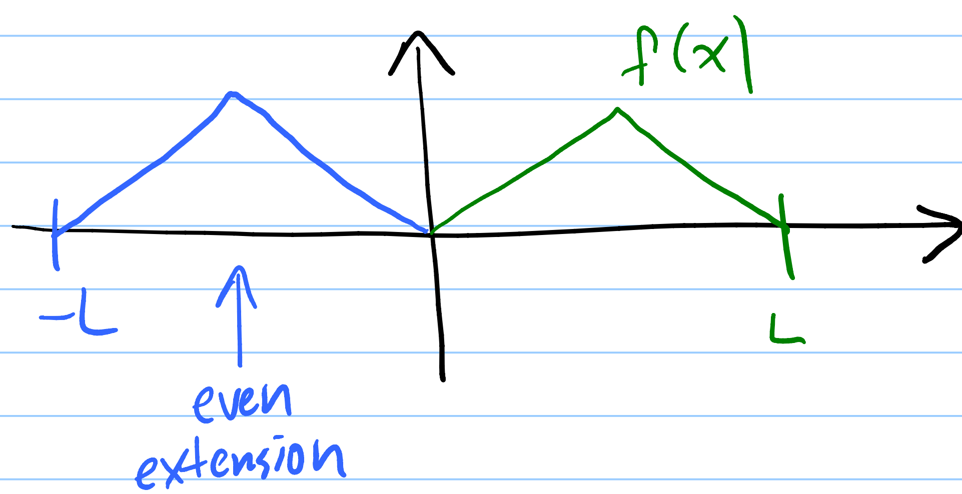
Sol^n: $$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L}$$

die away fast!

A Fourier Cosine Series!

I.C. $$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = f(x)$$

want



Full F.S. for even extension of f.

All b_n terms $= 0$.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

$= $ ave temp on $[0,L]$.

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}}\ \underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Fourier cosine series converges to f on $[0,L]$.

And we can read off that

$$\lim_{t\to\infty} u(x,t) = a_0 \leftarrow \text{ave initial temp on } [0,L]$$

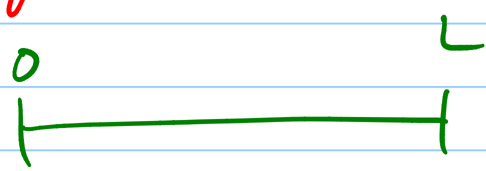
(Steady State sol^n)

Another famous problem:

Radiative boundary conditions.

(Rate of heat flow) prop to temp:

prop to temp gradient!



$$\underbrace{\frac{\partial u}{\partial x}(0,t)}_{\text{heat gushing out}} = \kappa\, u(0,t) \qquad \frac{\partial u}{\partial x}(L,t) = -\kappa\, u(L,t)$$

hot

Also, can mix BC's of type

1) homogeneous (refridgerated end)
2) insulated
3) radiative

New Sturm-Liouville Prob

$$X'' - \lambda X = 0$$

0 homog $X(0) = 0$

0 insulated $X'(0) = 0$

0 radiative $\bar{X}'(0) - k\bar{X}(0) = 0$

Sturm-Liouville Theory: Get orthoganal series of non-zero solutions.

Generalized Fourier Series.

Read Chapter 3.