Lecture 17 Parseval's Identity

Bessel's Ineq φ_i orthonormal on $[-\pi,\pi]$

$$\langle \varphi_i, \varphi_j \rangle = \int_{-\pi}^{\pi} \varphi_i(t)\overline{\varphi_j(t)}\,dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$f \sim \sum_{i=1}^{\infty} c_i \varphi_i \qquad c_i = \langle f, \varphi_i \rangle$$

Let $S_N = \sum_{i=1}^{N} c_i \varphi_i$ and $\sigma_N = \sum_{i=1}^{N} a_i \varphi_i$

arbitrary

$$\|f\|^2 = \int_{-\pi}^{\pi} |f|^2\,dt$$

$$0 \leq \underbrace{\left\| f - \sum_{i=1}^{N} c_i \varphi_i \right\|^2}_{\|f\|^2 - \sum_{i=1}^{N} |c_i|^2} \leq \left\| f - \sum_{i=1}^{N} a_i \varphi_i \right\|^2$$

Easy side. Bessel's always hold

Suppose f cont 2π-periodic. Fourier's Series. Fejér's yields σ_{N_0} with $\|f - \sigma_{N_0}\|^2 < \varepsilon$

Aha! $0 \leq \|f\|^2 - \sum_{1}^{N} |c_i|^2 \leq \varepsilon$ if $N > N_0$.

Bessel's shows $\sum_{1}^{\infty} |c_i|^2 \le \|f\|^2 < \infty$.

Fejér's yields these are equal! $\left(\begin{array}{c} f\ 2\pi \text{ periodic} \\ \text{cont.} \end{array} \right)$

Two versions of Parceval's:

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$$

Complex version:

$$\boxed{\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2\, dt}$$

Real version: $f \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

$$\boxed{2|a_0|^2 + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2\, dt}$$

Why: $\int_{-\pi}^{\pi} \underbrace{|e^{int}|^2}_{1} dt = 2\pi$

e^{int} orthogonal. $\quad \frac{e^{int}}{\sqrt{2\pi}} \leftarrow \varphi_n$ orthonormal

$$c_n = (f, \varphi_n) = \int_{-\pi}^{\pi} f(t) \frac{e^{-int}}{\sqrt{2\pi}}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$$

$$\sum c_n \varphi_n = \left(\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt \right) \frac{e^{int}}{\sqrt{2\pi}}$$

combine

Aha! Get regular F.S.

Parseval's $\sum |c_n|^2 = \int_{-\pi}^{\pi} |f(t)|^2\, dt$

but $c_n = \sqrt{2\pi}\, \hat{f}(n)$

$$2\pi \sum |\hat{f}(n)|^2 = \int_{-\pi}^{\pi} |f(t)|^2\, dt \quad \checkmark$$

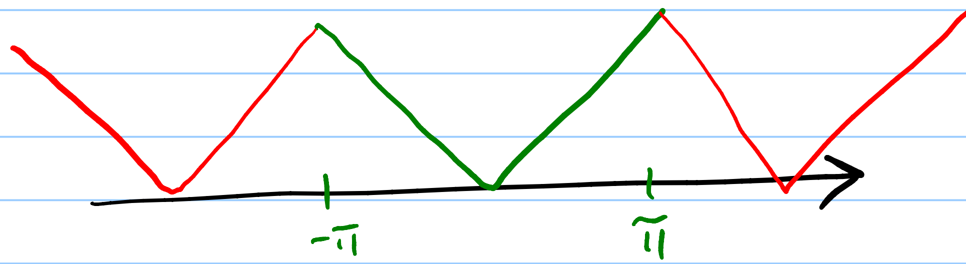
Orthonormalize: $1, \cos nt, \sin nt$

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos nt}{\sqrt{\pi}}, \quad \frac{\sin nt}{\sqrt{\pi}}$$

HWK 5: Chap 3: Exercise 8

Hints: Chap 2: Ex. 6: $f(t) = |t|$

$[-\pi, \pi]$



$$f(x) \sim \frac{\pi}{2} + \sum_{-\infty}^{\infty} c_n e^{int}$$

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{-2}{n^2} & n \text{ odd} \end{cases}$$

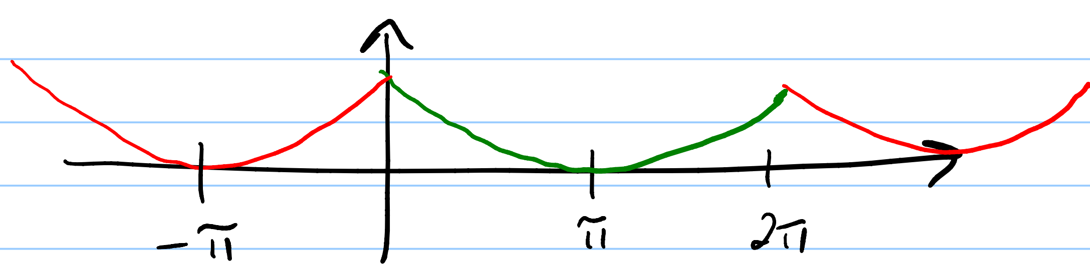
Parseval's will let you compute

$$\sum_{\substack{n \text{ odd} \\ n \geq 1}} \frac{1}{n^4}$$

Hmmm. Stein also wants $\sum_{1}^{\infty} \frac{1}{n^4}$.

Hint: Example 2 on p. 36

$f(t) = (\pi - t)^2/4$ on $[0, 2\pi]$.



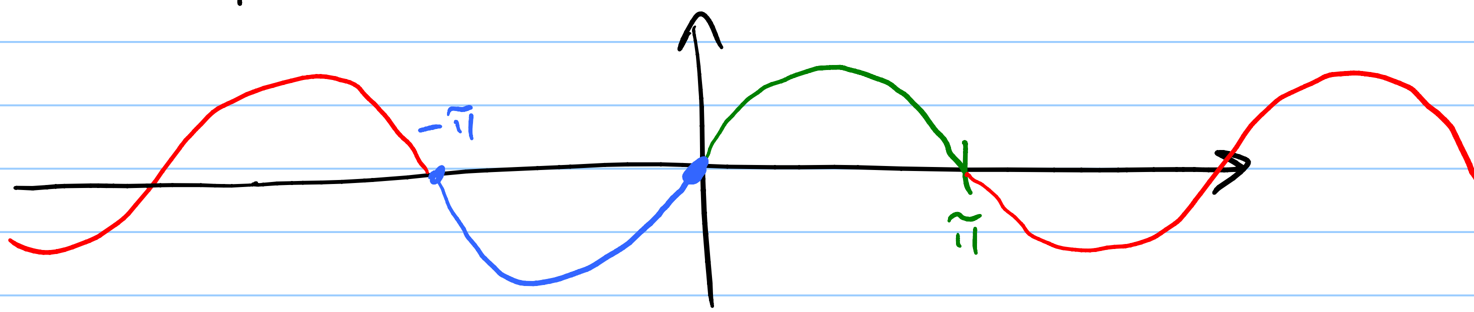
$$f \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nt$$

Use real version of Parseval's $[-\pi, \pi]$.

b) $f(t) = t(\pi - t)$ on $[0, \pi]$.

2π periodic odd extension

Chap 2: Ex. 4.



$$f \sim \frac{8}{\pi} \sum_{\substack{n \text{ odd} \\ \geq 1}} \frac{1}{n^3} \sin nt$$

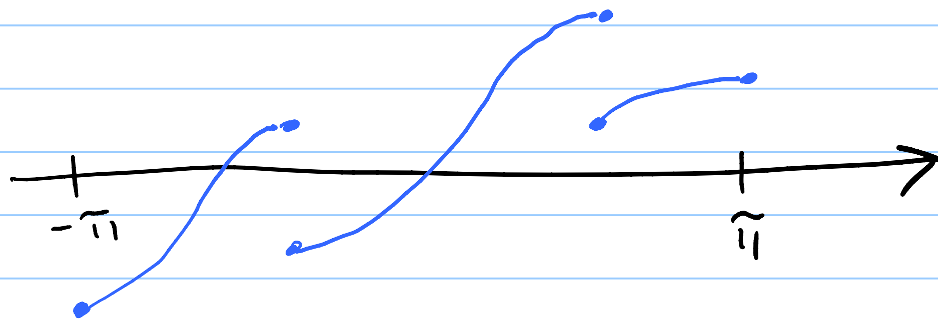
Real Parseval's yields $\displaystyle\sum_{\substack{n \text{ odd} \\ \geq 1}} \frac{1}{n^6}$

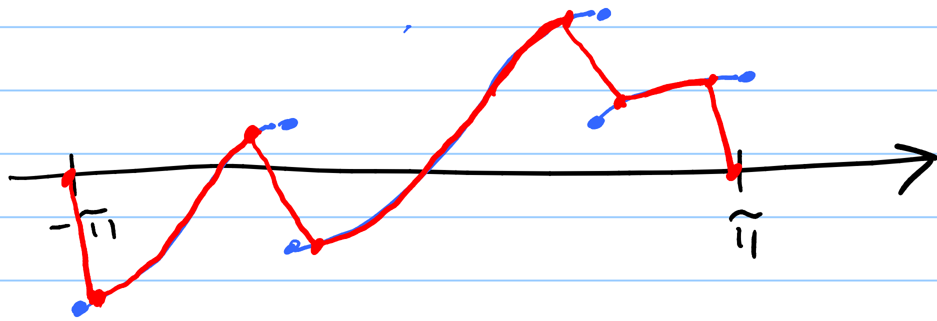
Stein: $\displaystyle\sum_{1}^{\infty} \frac{1}{n^6} = ?$ ⟵ won't grade!

Remark: $\displaystyle\sum^{\infty} \frac{1}{n^p}$ known for p even.

but $\displaystyle\sum_{1}^{\infty} \frac{1}{n^3}$ not known.

Parseval's Identity for piecewise continuous functions.

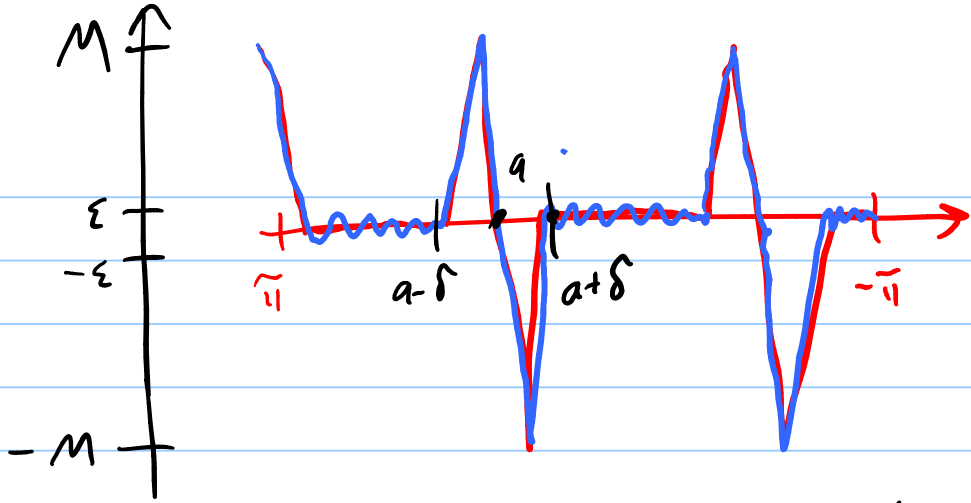




red − blue

Idea: Approximate cont 2π-periodic ext of red by $\sigma_N = \sum_{-N}^{N} a_n e^{int}$ given by Fejér.

$\sigma_N - (\text{blue})$ small in L^2 norm!



$$\|f-\sigma_N\|^2 \le \varepsilon^2\left(2\pi - n2\delta\right) + n2\delta M^2$$

$$\le \varepsilon^2 2\pi + n2\delta M^2$$

Pf of Parseval's

$$\underbrace{\|f - S_N\|}_{\|f\|^2 - \sum_{-N}^{N}|c_n|^2} \le \|f-\sigma_N\|^2 \le \left(\text{small!}\right)$$

Parseval's holds for piecewise continuous fcns!