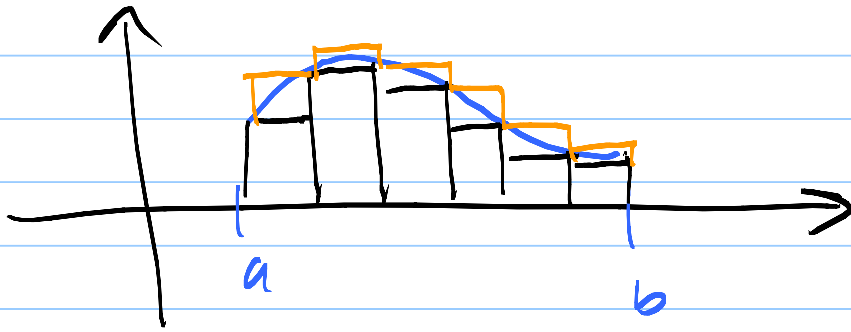
Lecture 18

Riemann Integral (Rudin p. 322)



Lower Sums $\leq \int_a^b f\,dx \leq$ Upper Sums

f Riemann Integral : Lower $\nearrow \int \swarrow$ Upper

Def^n: A set $S \subset \mathbb{R}$ is a set of measure zero, if, given $\varepsilon > 0$, $\exists$ seq of open inervals (a_n, b_n) such that

$$S \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$$

and
$$\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.$$

EX: $(0,1) \cap \mathbb{Q}$ is a set of measure zero.

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of them.

Let $\varepsilon > 0$. $r_n \in I_n$ with diam $< \frac{\varepsilon}{2^n}$.

Aha! $\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$

<u>Big Theorem</u>: f is R. Integrable on $[a,b]$ if and only if

1) f is bounded on $[a,b]$, and

2) f is continuous "almost everywhere" on $[a,b]$, meaning on $[a,b]$ minus a possible set of measure zero.

<u>Fact</u>: If f is R. Integrable, then

$$\mathcal{R}\int_a^b f\,dx = \mathcal{L}\int_a^b f\,dx$$

R. Integral

Lebesgue Integral

Uniqueness of Fourier Series

If f is piecewise continuous and $\hat{f}(n) = 0$ for all n, then $f \equiv 0$.

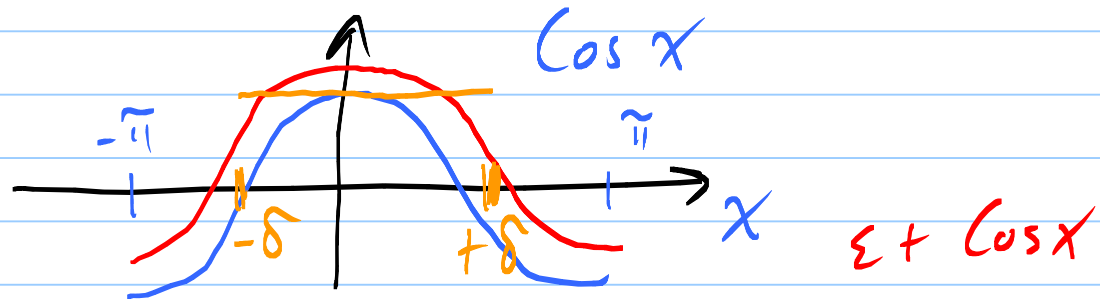
Pf: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = 0$

For piecewise cont. fcns $\int |f|^2 dx = 0$

$\Rightarrow f \equiv 0$. ✓

Stein: Suppose f Riemann Integrable and $\hat{f}(n) = 0$ for all n. Then $f \equiv 0$ on any interval where it is continuous.

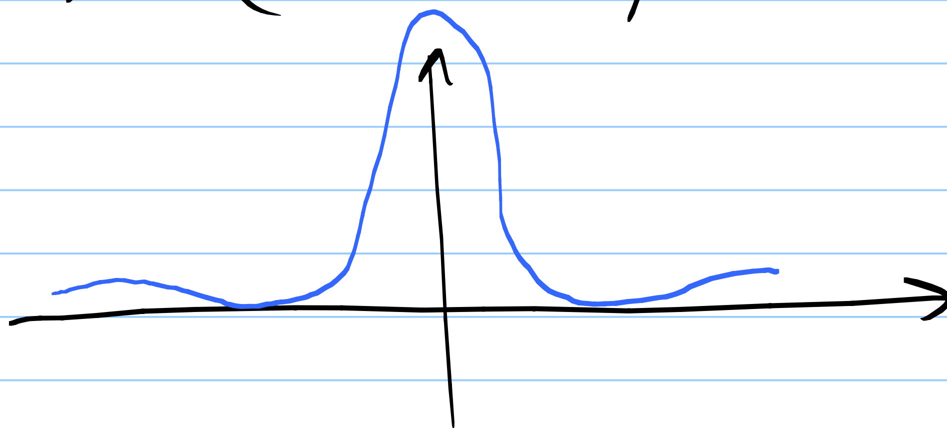
Pf:



By taking $\varepsilon > 0$ small, can make δ small $\quad \varepsilon + \cos x \geq 1$ in $[-\delta, \delta]$

$|\varepsilon + \cos x| < 1$ outside.

$f_n(x) = (\varepsilon + \cos x)^{2n}$ ← Trig poly!



$$f_n(x) = \left(\varepsilon + \frac{e^{ix} + e^{-ix}}{2}\right)^{2n}$$

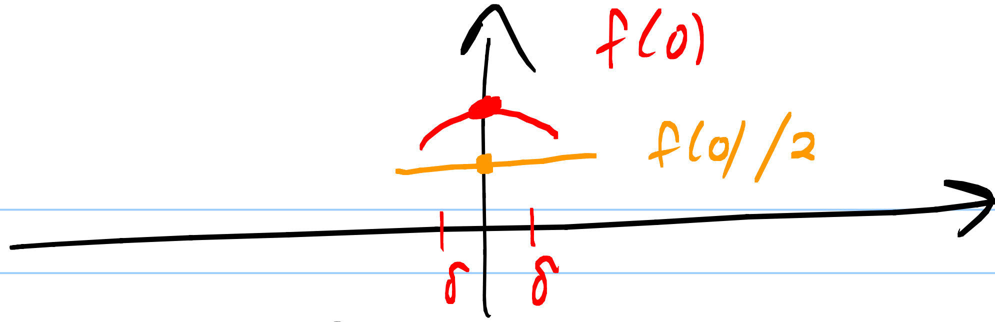
$$= \mathop{\mathrm{Sum}}_{k=-2n}^{2n} \left(c_k e^{ikx} + b_k e^{-ikx}\right)$$

$\hat{f}(n) = 0$ for all $n \Longrightarrow$

$f \perp$ (Trig polys)

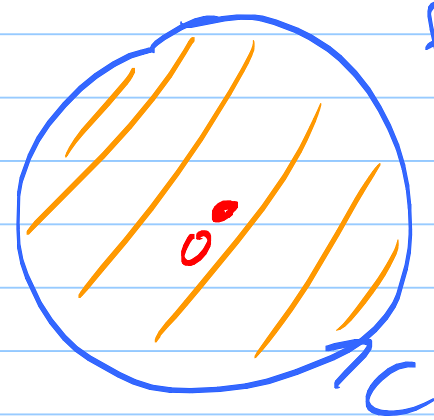
So $\displaystyle\int_{-\pi}^{\pi} f(x)\, f_n(x)\, dx = 0$

If f is cont on an interval about 0, suppose $f(0) \neq 0$. We can assume $f(0) > 0$.



Can see $\int_{-\pi}^{\pi} f\, f_n \, dx > 0$ for big n. ↯

Harmonic Functions:



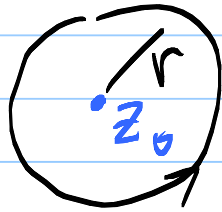
$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|re^{i\theta} - e^{i\psi}|} f(\psi)\, d\psi$$

u harmonic on $D_1(0)$, continuous up to C and $u(e^{i\psi}) = f(\psi)$.

Note: $u(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-0^2}{|0-e^{i\psi}|^2} f(\psi)\, d\psi$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\, d\psi \leftarrow \text{Ave of } f$$

<u>Fact</u>: Harmonic fcns satisfy the <u>averaging</u> property:



$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{i\psi}\right)d\psi$$

$$= \frac{1}{\underbrace{2\pi r}_{\text{arc length}}}\int_0^{2\pi} u\left(z_0 + re^{i\psi}\right)\underbrace{r\,d\psi}_{ds}$$

<u>Note</u>: Can diff'tiate under Poisson Int. So harmonic fcns are C^∞ diff'ble.

<u>Step 1</u>: Harmonic fcns satisfy the <u>maximum principle</u>: If u assumes a max inside $D_1(0)$, then $u \equiv$ const.

<u>Why</u>: Say u is max at $z_0 \in D_1(0)$

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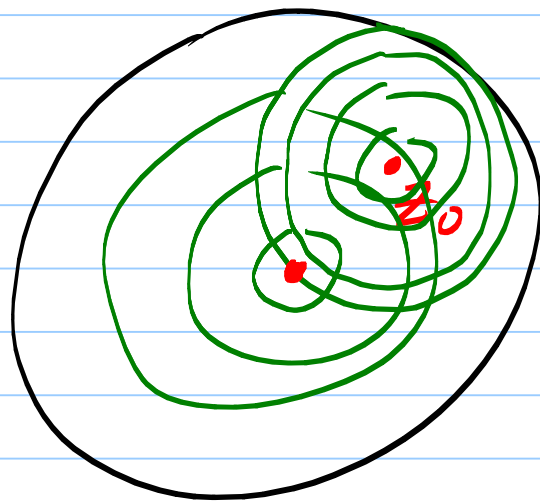

$D_1(0)$