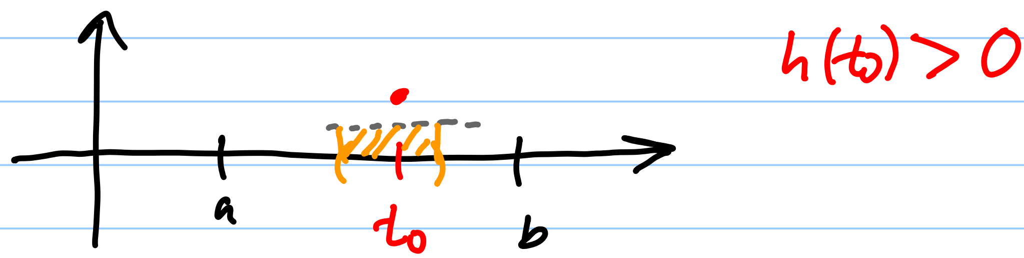
Lecture 19 Harmonic fcns

Calculus Lemmas:

1) Suppose $h(t) \geq 0$ and h continuous on $[a,b]$. Then

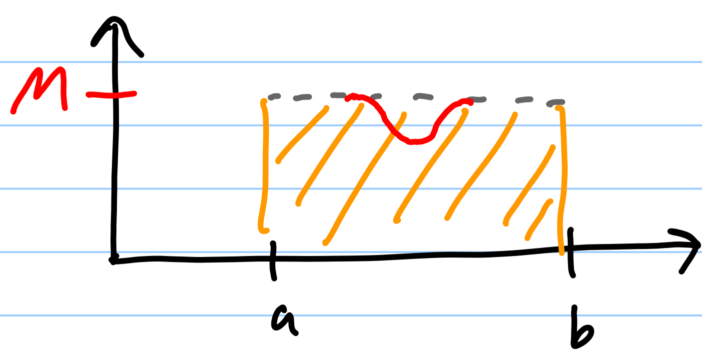
$$\int_a^b h\,dt = 0 \implies h \equiv 0.$$



2) Suppose $h(t) \leq M$, h cont..

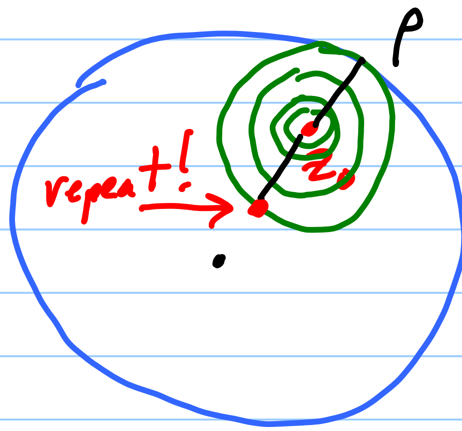
If $\int_a^b h(t)\,dt = M(b-a)$, then $h \equiv M$.

Pf: Apply #1 to $M - h(t)$.



<u>Thm</u>: u harmonic on $D_1(0)$ and continuous on $\overline{D_1(0)}$. If u assumes its max M at $z_0 \in D_1(0)$, then $u \equiv M$ on $\overline{D_1(0)}$. [Maximum Princ.]

<u>Pf</u>:



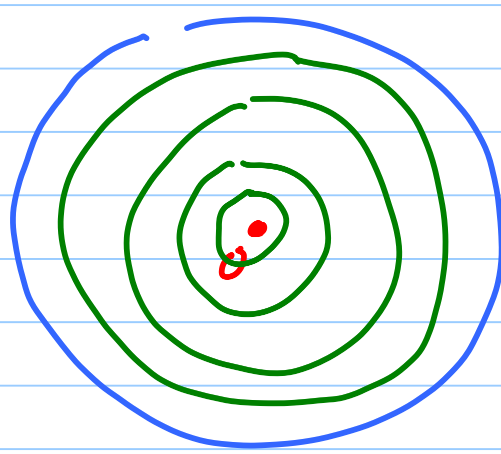
$u(z_0) = M$, the max

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + re^{it})}_{\le M}\, dt$$

Calc Lemma #2 $\Rightarrow$ $u \equiv M$ on green circles.

Get bigger and bigger circles: Radius $= \rho, 2\rho, 4\rho, \ldots$

Eventually, get the origin inside green circle.



$u \equiv M$ on $D_1(0)$

<u>Cor</u>: Solution to D. Prob is unique.

Pf: Two sol^{n}s u_1, u_2

$$\Delta u_j \equiv 0, \quad u_j(e^{i\theta}) = f(\theta).$$

$u_1 - u_2$ is harmonic on $D_1(0)$, $\equiv 0$ on $C_1(0)$.

$u_1 - u_2 \leq$ Max on $C_1(0) \leftarrow \equiv 0$

Same $u_2 - u_1$. Get $u_1 \equiv u_2$.

Cor of Max Princ: Harmonic fcns attain their Maxes on the boundary.

Pf: Know a continuous on $\overline{D_1(0)}$ attains its max, say at z_0. If $z_0 \in D_1(0)$, Max P. $\Rightarrow u \equiv$ const. $\Rightarrow u$ attains max on $C_1(0)$.

Case $z_0 \in$ bndry: u attains max on bndry. ✓

Theorem: A continuous fcn that satisfies the averaging property must be harmonic!

Pf: Suppose u is cont on $\overline{D_1(0)}$ and satisfies ave prop.

Let $\tilde{u}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|re^{i\theta}-e^{i\psi}|^2} u(e^{i\psi})\,d\psi.$

We know $\tilde{u}$ is harmonic on $D_1(0)$ and continuous on $\overline{D_1(0)}$ and $\tilde{u}(e^{i\theta}) = u(e^{i\theta})$.

$\tilde{u}-u$ satisfies the ave prop, which implies the Max Princ.

So $\tilde{u}-u$ attains its max on $C_1(0)$.

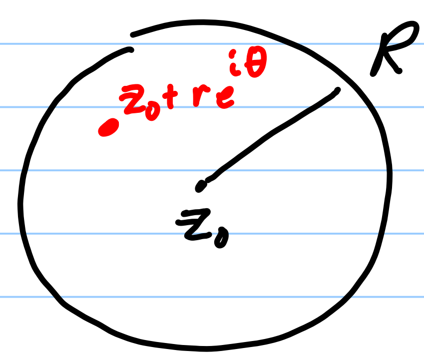
But it is zero on $C_1(0)$. So $\tilde{u}-u \leq 0$ on $\overline{D_1(0)}$. Repeat for $u-\tilde{u}$. Get $u-\tilde{u} \leq 0$.

So $u \equiv \tilde{u} \leftarrow$ harmonic!

Liouville's Theorem for harmonic fcns on $\mathbb{R}^2$.

If u harmonic on $\mathbb{R}^2$ and bounded there, then $u \equiv$ const on $\mathbb{R}^2$.

Pf:



$$u(z_0+re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-\psi)+r^2}\, u(z_0+Re^{i\psi})\, d\psi$$

Take $r=x$, $\theta=0$.

$$\left|\frac{\partial u}{\partial x}(z_0)\right| = \frac{1}{2\pi}\int_0^{2\pi} \left| \underbrace{\frac{\partial}{\partial x}\left[\;\sim\sim\sim\;\right]}_{\leq \frac{1}{R}} \;\underbrace{u(z_0+Re^{i\psi})}_{\leq M} \right| d\psi$$

$$\left|\frac{\partial u}{\partial x}(z_0)\right| \leq \frac{M}{R} \qquad (A)$$

Take $r=y$, $\theta=\frac{\pi}{2}$. Repeat,

$$\left|\frac{\partial u}{\partial y}(z_0)\right| \leq \frac{M}{R} \qquad (B)$$

Aha! If u is harmonic on $\mathbb{R}^2$ and $|u|\leq M$ on $\mathbb{R}^2$, pick $z_0\in\mathbb{R}^2$.

Let $R>0$. (Think big!).

(A) and (B) yield

$$|\nabla u(z_0)| \leq \sqrt{2}\,\frac{M}{R}$$

Let $R \to \infty$. See $\nabla u(z_0)$ must be zero. z_0 arbitrary! So $\nabla u \equiv 0$ on $\mathbb{R}^2$.

$\Longrightarrow$ u const.

$$\left[\int_{\gamma_A^B} \nabla u \cdot d\vec{x} = u(B) - u(A) \right]$$

Fun Fact: Liouville's Thm $\Longrightarrow$

Fund. Thm. Alg.

Analytic functions: $f(z) = z^2 \quad z \in \mathbb{C}$

$$DQ = \frac{f(z) - f(a)}{z - a} = \frac{z^2 - a^2}{z - a} = \frac{\cancel{(z-a)}(z+a)}{\cancel{z-a}}$$

$$= z + a \longrightarrow a + a = 2a \quad \text{as } z \to a.$$

$f(z)=z^2$ is complex diff'ble and $f'(z)=2z$.

<u>Def^n</u>: If f is complex diff'ble on an open set Ω, we say f is <u>analytic</u> on Ω.

<u>Fact</u>: Complex polys are analytic.
Complex rational f'cns are analytic where denom $\neq 0$.

<u>Fact</u>: Real and imaginary parts of analytic f'cns are harmonic.