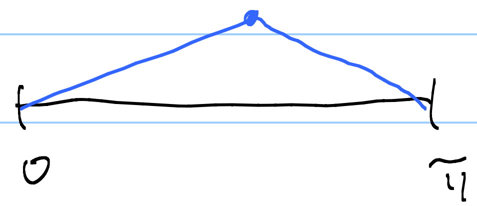
Lecture 2: Harpsichord prob:



$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x^2}$$

Johann: Try $u(x,t) = X(x)T(t)$

$$\frac{X''}{X} = \frac{T''}{T} = \lambda$$

$$\boxed{\begin{gathered} X'' - \lambda X = 0 \\ X(0) = 0,\ X(\pi) = 0 \end{gathered}}$$

Must have $\lambda = -n^2$ in order to get non-zero sol^{n}s.

For $\lambda = -n^2$, got $X_n(x) = \sin nx$

T-problem: $T'' - \lambda T = 0$ without B.C.'s.

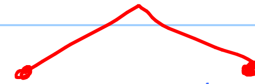
$\lambda = -n^2$: $T(t) = A_n \cos nt + B_n \sin nt$

Find $u_n(x,t) = \sin nx \left[A_n \cos nt + B_n \sin nt \right]$

Try $u(x,t) = \sum_{n=1}^{\infty} \sin nx \left[A_n \cos nt + B_n \sin nt \right]$

Last things:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx \underbrace{\cos n0}_{1} \overset{\text{want}}{=} f(x)$$



$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left(-nA_n \sin nx \sin nt + nB_n \sin nx \cos nt \right)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} nB_n \sin nx \overset{\text{want}}{=} 0$$

Aha! Take all B_n's $= 0$.

Sol^n: $$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \cos nt}$$

Big problem: Given $f(x)$, can we find A_n's so that $\sum_{n=1}^{\infty} A_n \sin nx = f(x)$?

Key: $\sin nx$ are orthogonal on $[0, \pi]$.

$$\langle \sin nx, \sin mx \rangle = \int_0^{\pi} \sin nx \sin mx \, dx$$

$$= \begin{cases} 0 & n \neq m \\ \int_0^{\pi} \sin^2 nx \, dx = \end{cases}$$

$$\int_0^{\pi} \frac{1}{2}\left(1 - \cos 2nx\right) dx = \frac{\pi}{2}$$

$$\int_0^\pi \sin nx \cdot f(x)\,dx = \int_0^\pi \left[A_1 \sin x + A_2 \sin 2x + \cdots\right] \cdot \sin nx\,dx$$

$$= \sum_{m=1}^{\infty} A_m \int_0^\pi \sin mx \sin nx\,dx$$

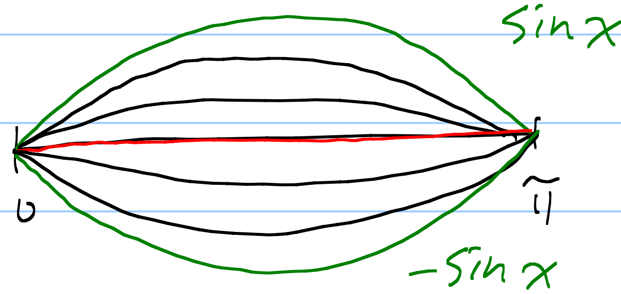
$$= A_n \underbrace{\int_0^\pi \sin^2 nx\,dx}_{\pi/2}$$

Aha! $$\boxed{A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx.}$$

Modes of oscilation:

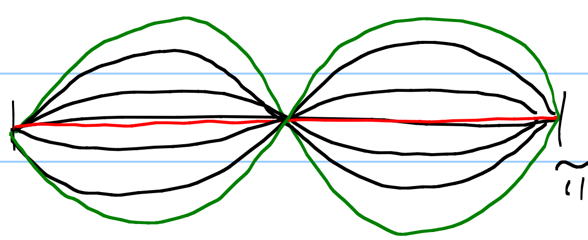
$$u_n(x,t) = \sin nx \cos nt$$

$n=1$



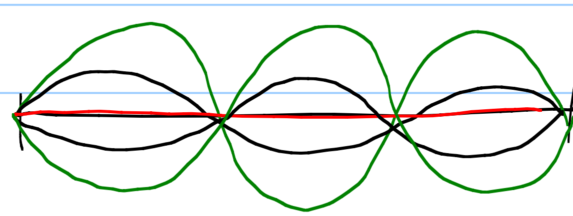
$\sin x \cos t$

$n=2$

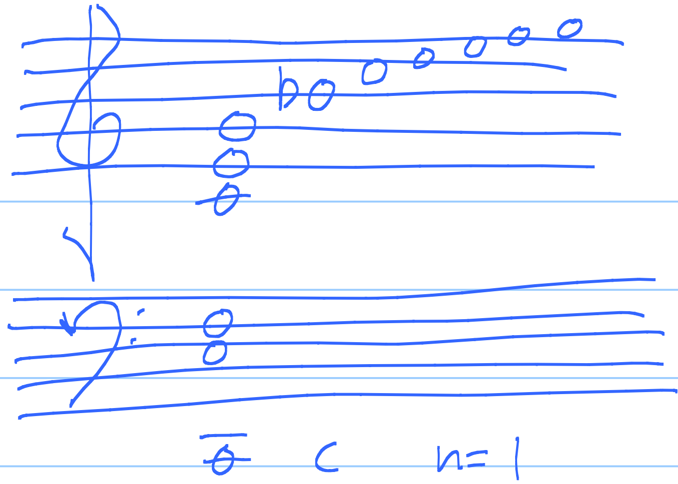


$\sin 2x \cos 2t$

$n=3$



$\sin 3x \cos 3t$



G
E
C $n=4$

G $n=3$

C $n=2$

C $n=1$

$$A_1 = \frac{1}{1^2}, \quad A_2 = 0, \quad A_3 = -\frac{1}{3^2}, \quad A_4 = 0, \quad A_5 = \frac{1}{5^2}, \quad \ldots$$

Harpsichord sol^n:

$$u(x,t) = \sin x \cos t - \frac{1}{3^2} \sin 3x \cos 3t + \frac{1}{5^2} \sin 5x \cos 5t - \cdots$$