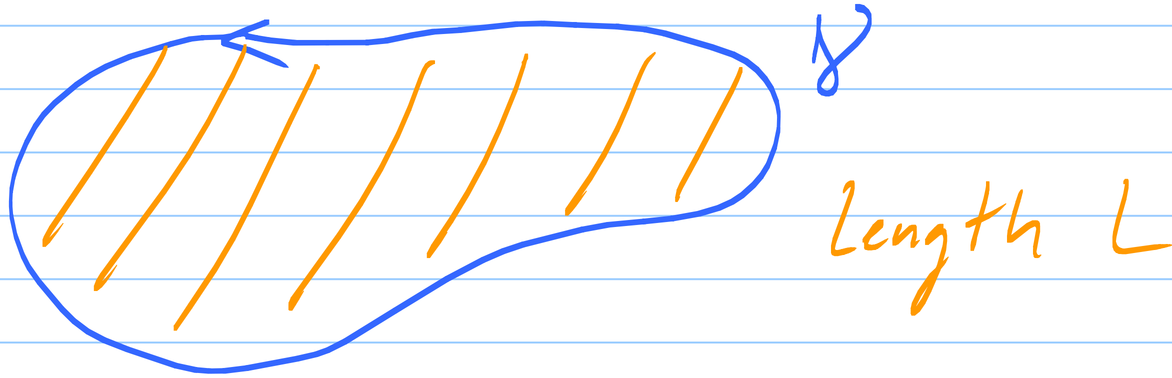
Lecture 22 The Isoparametric Ineq.



Area Circle $A = \pi r^2$ $(L = 2\pi r)$

$$A = \pi \left(\frac{L}{2\pi}\right)^2 \qquad r = \frac{L}{2\pi}$$

$$A = \frac{L^2}{4\pi}$$

Non-circle: $\boxed{A < \frac{L^2}{4\pi}}$

Consequences of Parseval's Identity:

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dt$$

$c_n = \hat{f}(n)$

$$\sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$$

$$f = \underbrace{\sum_{-N}^{N} \hat{f}(n)e^{int}}_{f_N} + R_f$$

remainder

$$\|R_f\|^2 = \frac{1}{2\pi}\underbrace{\sum_{|n|>N} |\hat{f}(n)|^2}_{\to 0 \text{ as } N\to\infty}$$

$$\langle f, g\rangle = \langle f_N + R_f, g_N + R_g\rangle$$

$$= \underbrace{\langle f_N, g_N\rangle}_{\sum_{-N}^{N} \hat{f}(n)\hat{g}(n)} + \text{stuff} \to 0 \text{ in } L^2$$

<u>Cauchy-Schwarz Ineq:</u>

$$\left|\sum a_n \bar{b}_n\right| \le \left(\sum |a_n|^2\right)^{1/2}\left(\sum |b_n|^2\right)^{1/2}$$

$$\left| \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt \right| \le \left(\int_{-\pi}^{\pi} |f|^2 dt \right)^{1/2} \left(\int_{-\pi}^{\pi} |g|^2 dt \right)$$

Fourier Series : $f \longrightarrow \hat{f}(n)$

$$L^2(-\pi, \pi) \longleftrightarrow \ell^2$$

Pf of Isop. Ineq : Assume Length γ $= L = 2\pi$. [Think $r = 1$.]

Want to show $A \le \pi$.

Green's Thm :

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Aha! Let $Q = x$, $P = -y$.

$$\boxed{\int_\gamma -y\,dx + x\,dy = 2\iint_\Omega 1\,dA = 2A}$$

Important step: Parametrize γ by arc length t:

$$\gamma : \begin{cases} x(t) = \sum_{-\infty}^{\infty} \hat{x}(n)\, e^{int} \\ y(t) = \sum_{-\infty}^{\infty} \hat{y}(n)\, e^{int} \end{cases}$$

$-\pi \leq t \leq \pi$

t arc length. So $\dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}$ has <u>unit</u> length. $\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} = 1$

So $\underline{\dot{x}(t)^2 + \dot{y}(t)^2 = 1}$ too!

$$A = \frac{1}{2}\int_\gamma -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} -y(t)\,\dot{x}(t)\,dt + x(t)\,\dot{y}(t)\,dt$$

Assume γ is C^2-smooth.

Know: $\hat{\dot{x}}(n) = in\,\hat{x}(n)$

$$A = \frac{1}{2}\left[2\pi \sum_{-\infty}^{\infty}\left(-\hat{y}(n)\,\overline{[in\,\hat{x}(n)]} + \hat{x}(n)\,\overline{[in\,\hat{y}(n)]}\right)\right.$$

$$\underline{A = \frac{1}{2}\cdot 2\pi \sum_{-\infty}^{\infty} -in\left(\hat{x}(n)\overline{\hat{y}(n)} - \hat{y}(n)\overline{\hat{x}(n)}\right)}$$

Next $\displaystyle\int_{-\pi}^{\pi} \underbrace{\dot{x}(t)^2 + \dot{y}(t)^2}_{1}\,dt = 2\pi$

Parseval's: $\displaystyle 2\pi \sum_{n=-\infty}^{\infty}\left(|in\,\hat{x}(n)|^2 + |in\,\hat{y}(n)|^2\right) = 2\pi$

$$\underline{\pi = \pi \sum_{n=-\infty}^{\infty} n^2\left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2\right)}$$

$$\pi - A =$$

$$\pi \sum_{n=-\infty}^{\infty} \left[n^2 \left(|\hat{x}(n)|^2 + |\hat{y}(n)|^2 \right) + in \left(\hat{y}(n)\overline{\hat{x}(n)} - \hat{x}(n)\overline{\hat{y}(n)} \right) \right]$$

Write $\hat{x}(n) = \alpha_n + i\beta_n$

$\hat{y}(n) = a_n + i\, b_n$

Algebra:

$$\pi - A = \pi \sum_{-\infty}^{\infty} \left[n^2 \left(\alpha_n^2 + \beta_n^2 + a_n^2 + b_n^2 \right) - 2n \left(\beta_n \alpha_n - \alpha_n \beta_n \right) \right]$$

$$= \pi \sum_{\substack{-\infty \\ n \neq 0}}^{\infty} \left[(n\alpha_n + b_n)^2 + (n\beta_n - a_n)^2 + (n^2-1)(a_n^2 + b_n^2) \right]$$

Aha! A sum of terms ≥ 0!

So $\pi - A \geq 0$. QED!

Furthermore, if $\tilde{\pi} - A = 0$, we see that $a_n = b_n = \alpha_n = \beta_n = 0$ for $n = \pm 2, \pm 3, \pm 4, \ldots$

For $n = \pm 1$, we have

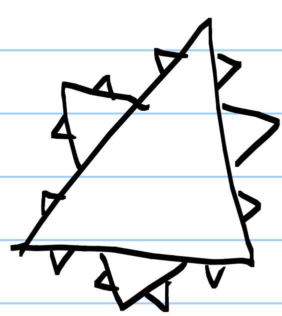
$$\alpha_{\pm 1} = \mp b_{\pm 1} \qquad \beta_{\pm 1} = \pm a_{\pm 1}$$

So

$$\begin{cases} x(t) = \hat{x}(0) + 2\alpha_1 \cos t - 2\beta_1 \sin t \\ y(t) = \hat{y}(0) + 2\beta_1 \cos t + 2\alpha_1 \sin t \end{cases}$$

Can show this is a circle!

Green's Thm: Proved in Spivak: Calculus on Manifolds.



Area stays finite. Perem $\to \infty$.