# Lecture 23 Dirichlet Problem with polynomial boundary data on $D_1(0) = \{z : |z| < 1\}$

$$C_1(0) = \{z : |z| = 1\}.$$

Schwarz fcn for $C_1(0)$: $S(z) = 1/z$

On $C_1(0)$: $z = e^{i\theta}$ $\quad \frac{1}{z} = e^{-i\theta} = \bar{z}$

Schwarz fcn: $S(z) = \bar{z}$ on $C_1(0)$.

Observation: Real poly $p(x,y) = \sum c_{ij} x^i y^j$

Degree $p = \text{Max}(i+j)$ : $c_{ij} \neq 0$ $\quad (z = x + iy)$

$$x = \frac{1}{2}(z + \bar{z})$$

$$y = \frac{1}{2i}(z - \bar{z})$$

Get $p(x,y) = \sum a_{nm} z^n \bar{z}^m$ ← poly in $z$ and $\bar{z}$

**Trick:** $n > m$: $z^n \bar{z}^m = z^{n-m} \underbrace{(z^m \bar{z}^m)}_{(z\bar{z})^m = 1 \text{ on } C_1(0)}$ $\quad z\bar{z} = |z|^2 = 1$ on $C_1(0)$

$$= z^{n-m} \text{ on } C_1(0)$$

$n < m$: $z^n \bar{z}^m = \bar{z}^{m-n}(\bar{z}^n z^n) = \bar{z}^{m-n}$ on $C_1(0)$

$n = m$: $z^n \bar{z}^n = 1$ on $C_1(0)$.

Aha! $p(x,y) = P(z,\bar z) = \underbrace{\left( \sum A_n z^n + B_m \bar z^m \right)}_{\text{on } C(0)} + C$

harmonic and solves D. Prob for $p(x,y)$! Unique sol$^n$!

Property of $D_1(0)$: Sol$^n$ to D. Prob with polynomial boundary data $\varphi$ is polynomial.

Application of Fejér's Thm:

Suppose $\varphi$ is $2\pi$-periodic and continuous.

Fejér's Thm says, given $\varepsilon = \frac{1}{n}$, there is a trig poly $P = \sum_{-N}^{N} c_n e^{in\theta}$ so that

$|\varphi - P| < \frac{1}{n}$

Hmmm.

$$\underbrace{\begin{matrix} z = e^{i\theta} = x+iy & \quad & \bar z = e^{-i\theta} = x - iy \\ z^n = e^{in\theta} & \quad & \bar z^n = e^{-in\theta} \end{matrix}}_{\text{on } C_1(0)}$$

Aha! See harmonic ext of $P_N = \sum_{n\geq 0} c_n z^n + \sum_{n>0} c_{-n} \bar z^n$

Big Idea: Get $P_n$ harmonic

$$|\varphi - P_n| < \frac{1}{n} \quad \text{on} \quad C_1(0).$$

$P_n \to \varphi$ unif on $C_1(0)$.

Hence they are uniformly Cauchy on $C_1(0)$.

Unif Cauchy estimate $|P_n - P_m| < \varepsilon$

extends to hold on $\overline{D_1(0)}$ by Max Princ.

So $P_n$ converge uniformly on $\overline{D_1(0)}$

to a continuous fcn $u$.

Claim: $u$ solves D. prob for boundary data $\varphi$.

1) $u$ continuous ✓

2) $u(e^{i\theta}) = \varphi(\theta)$ ✓

3) $u$ satisfies the averaging property

and is therefore harmonic!

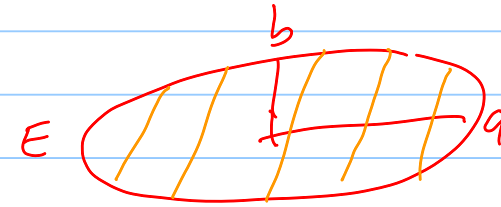
$$u(z_0) = \lim P_n(z_0) = \lim \int_0^{2\pi} P_n(z_0 + re^{it})\,dt$$

$$\underbrace{=}_{\text{because of unif conv.}} \int \lim = \int_0^{2\pi} u(z_0 + re^{it})\,dt \ ✓$$

What domains in plane have the poly $\longrightarrow$ poly sol$^n$ for D. Prob?

EX: Ellipse : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$r(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$




$$E : \{ (x,y) : r(x,y) < 0 \}$$

defining fcn for Ellipse.

Fisher Map : $\Delta(r p)$ = a new poly

poly

$\Delta$ is 2$^{nd}$ order linear. Reduces deg by 2.

$r$ is 2$^{nd}$ degree. $rp$ increases deg by 2.

$$\tilde{T} p = \Delta(rp).$$

Let $\mathcal{P}_N$ = N-th degree polys.

$$\tilde{T} : \mathcal{P}_N \longrightarrow \mathcal{P}_N .$$

finite dim$^l$ vect. space

$\tilde{\mathcal{F}}$ is a linear transf.

Claim: $\tilde{\mathcal{F}}$ is 1-1.

Why: Suppose $\tilde{\mathcal{F}}(p) = 0$.

Need to show $p \equiv 0$.

$$\Delta(\underbrace{rp}_{\text{harmonic}}) = 0$$

But $rp$ vanishes on boundary.

Max Princ $\Longrightarrow$ $rp \equiv 0$ on $E$.

But $r<0$ inside $E$. So $p \equiv 0$ on $E$. ✓

Lin. Alg. Fact: $V$ fin. dim. vector space.

$\tilde{\mathcal{F}}: V \to V$ is 1-1.

Then $\tilde{\mathcal{F}}$ is onto!

D. Prob on $E$: Suppose $p$ is a poly.

$\Delta p$ is a poly too. Fisher map is onto.

So there is a poly $q$ with $\Delta(rq) = \Delta p$.

Aha! $p - rq$ solves the D. Prob with data $p$.

Why: $\Delta(\underbrace{p-rq}_{=p \text{ on bndry because } r=0 \text{ on bndry.}}) = \Delta p - \Delta(rq) = 0$

$p - rq$ is a poly!

Khavinson-Shapiro Conjecture: A domain with poly $\to$ poly D. Prob property must be a disc or an ellipse.

Still open!

but (B., Ebenfelt, Khavinson, Shapiro)

If a domain has the property that rational boundary data leads to rational sol$^n$s to the D. Prob, then it must be a disc.