Lecture 24 Solutions to Midterm.

1. x^7, $\cos 3x$, 1 $\perp$ on $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{x^7} \; \underset{\text{even}}{\cos 3x}}_{\text{odd}} \, dx = 0$$

$$\int_{-\pi}^{\pi} \underbrace{x^7 \cdot 1}_{\text{odd}} \, dx = 0$$

$$\int_{-\pi}^{\pi} \underbrace{1 \cdot \cos 3x}_{\text{even}} \, dx = 2\int_{0}^{\pi} \cos 3x \, dx$$

$$= 2\left[\frac{1}{3}\sin 3x\right]_0^{\pi} = 0$$

If $x^7 f(x) = \left[c_0 + c_1 x^7 + c_2 \cos 3x\right] x^7$

$$\int_{-\pi}^{\pi} x^7 f(x)\, dx = c_1 \int_{-\pi}^{\pi} (x^7)^2 \, dx$$

$$= c_1 \left[\frac{1}{15} x^{15} \right]_{-\pi}^{\pi}$$

$$c_1 = \frac{\int_{-\pi}^{\pi} x^7 f(x)\, dx}{\left(\frac{2}{15} \pi^{15} \right)}$$

2. $X'' + \lambda X = 0$ on $[0, \pi]$

$X'(0) = 0$, $X(\pi) = 0$

$\underline{\lambda > 0}$: $\lambda = k^2$

$$r^2 + k^2 = 0$$

$$r = \pm ki$$

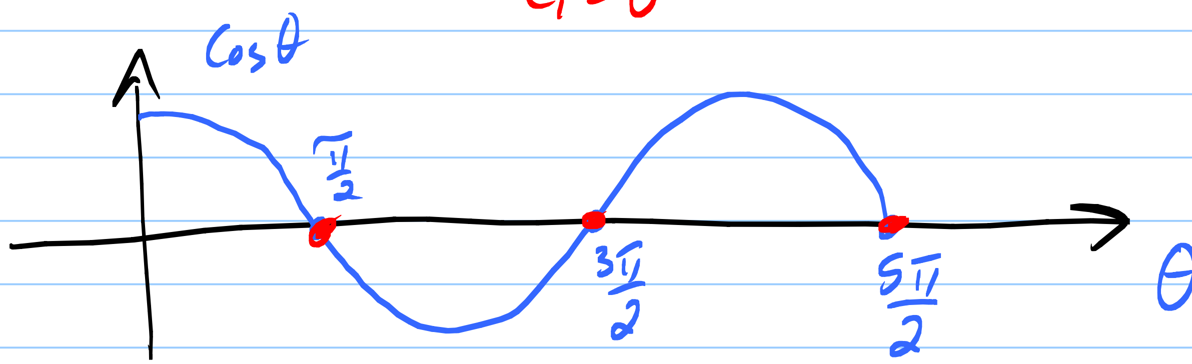
$$\underline{X}(x) = \underline{c_1 \cos kx} + \cancel{c_2 \sin kx}$$

$$\underline{X}'(x) = -c_1 k \sin kx + \cancel{c_2 k \cos kx}$$

$\underline{\text{Need}}$: $X'(0) = 0 + c_2 k \cdot 1 \overset{\text{want}}{=} 0$

$$\boxed{c_2 = 0}$$

and $$X(\pi) = \underset{\substack{\uparrow \\ \text{don't want} \\ c_1 = 0}}{c_1} \underbrace{\cos k\pi}_{\text{need this} = 0.} \overset{\text{want}}{=} 0$$



Need $$k\pi = (\text{odd})\frac{\pi}{2} \overset{\text{or}}{=} (n+\tfrac{1}{2})\pi$$
$n = 0, 1, 2, \ldots$

So $\lambda = k^2 = \left[(n+\frac{1}{2})\right]^2$, $n = 0, 1, 2, 3, \ldots$

For n: $$X_n = \underbrace{\cos(n+\tfrac{1}{2})x}_{\text{eigenfunctions for } \frac{d^2}{dx^2}}$$

$$X'' = -\lambda X$$

Orthogonal and complete! Parseval's Identity.

3. $(1-z)(1+z+z^2+\cdots+z^N) =$

$$
\begin{array}{rl}
 & 1 + z + \cdots + z^N \\
- & \quad\;\; z + z^2 + \cdots + z^N + z^{N+1} \\
\hline
 & 1 - z^{N+1}
\end{array}
$$

$$1+z+\cdots+z^N = \frac{1-z^{N+1}}{1-z}$$

$$\cos\theta = \operatorname{Re} e^{i\theta}$$

$$\cos n\theta = \operatorname{Re} e^{in\theta}$$

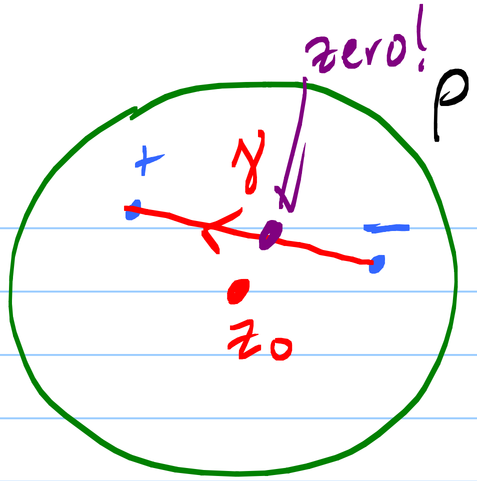
So $1 + \cos\theta + \cdots + \cos N\theta$

$$= \operatorname{Re}\left[1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{iN\theta}\right]$$

$$= \operatorname{Re}\left[1 + (e^{i\theta}) + (e^{i\theta})^2 + \cdots + (e^{i\theta})^N\right]$$

$$= \operatorname{Re}\left[\frac{1-e^{i(N+1)\theta}}{1-e^{i\theta}}\right]$$

4.



z_0 isolated zero of u

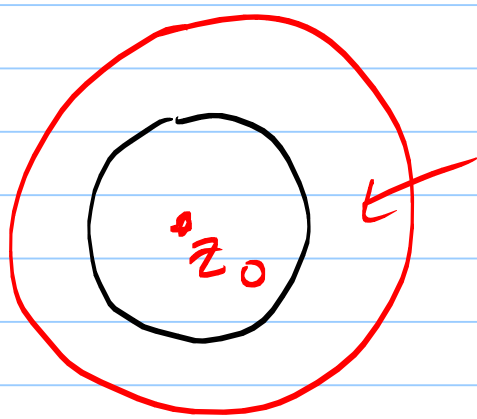
Shrink ρ so it is the only zero on $D_\rho(z_0)$.

Can make γ miss z_0. $\gamma: \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$0 \le t \le 1$. $\underbrace{u(x(t), y(t))}_{h(t)}$ continuous.

$h(0) < 0$, $h(1) > 0$. I.V.T. gives

t_0, $0 < t_0 < 1$ where $h(t_0) = 0$. ↯



always + on $D_\rho(z_0) - \{z_0\}$.

[or take $-u$ in place of u]

$$0 = u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + re^{it})}_{>0}\, dt > 0$$

Rational solutions to the Dirichlet Problem with rational data on the unit disc.

Key: Schwarz function for $C_1(0)$:

$$S(z) = \frac{1}{z} = \bar{z} \text{ on } C_1(0)$$

$$\left[\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \bar{z} \text{ on } C_1(0). \right]$$

Data $\varphi(x,y) = R(x,y)$ rational.

$$= R\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

$$= Q(z, \bar{z}) \text{ rational in } z \text{ and } \bar{z}.$$

On C_1: $= Q\left(z, \frac{1}{z}\right)$

$= q(z) \leftarrow$ rational in z

= Partial Fractions Decomp

$$\sum_{n=1}^{N} \sum_{k=1}^{M_n} \frac{A_{nk}}{(z-a_n)^k} + \underbrace{P(z)}_{\text{poly in } z}$$

$$= \underbrace{\sum_{a_n \text{ inside } D_1(0)}} + \underbrace{\sum_{a_n \text{ outside}} + P(z)}_{\text{analytic in } z \text{ on } D_1(0)}$$

$$\underbrace{\sum \sum \frac{A_{nk}}{(\frac{1}{\bar z} - a_n)^k}}_{= \text{Conjugate of an analytic fcn of } z.} \leftarrow \text{blows up at } z = \frac{1}{\bar a_n} \text{ \underline{outside} } C_1(0)!$$

$$P(x,y) = \overline{h(z)} + H(z) \quad \text{on } C_1$$

where h and H are

rational and analytic
on a disc $D_\rho(0)$ with $\rho > 1$.

But this extends to $\overline{D_1(0)}$
and solves the D. Prob.

<u>Thm</u>: Discs are the <u>only</u> domains
in the plane with rational data
$\longrightarrow$ rational sol^n property.