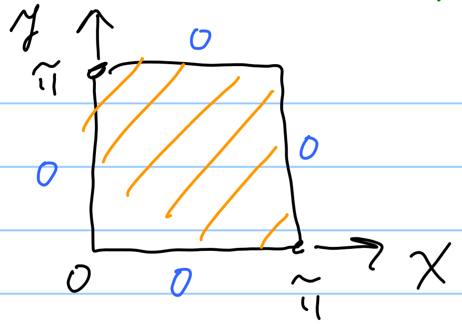
Lecture 25 Square drum



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$$u(x,y,t) = X(x)Y(y)T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = XYT'' = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^2\left[X''YT + XY''T\right]$$

Divide by XYT:

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

Save T for last.

$$\underset{\substack{\uparrow \\ \text{fcn of } x}}{\frac{X''}{X}} = \underbrace{\lambda - \frac{Y''}{Y}}_{\substack{\uparrow \\ \text{fcn of } y}} = \mu \leftarrow \text{a const}$$

Two problems:

$$X'' - \mu X = 0$$

$$\begin{cases} X(0) = 0 & \text{left} \\ X(\pi) = 0 & \text{right} \end{cases}$$

Need $\boxed{\mu = -n^2}$

$$X_n(x) = \sin nx$$

$$Y'' - (\lambda - \mu) Y = 0$$

$$\begin{cases} Y(0) = 0 & \text{bottom} \\ Y(\pi) = 0 & \text{top} \end{cases}$$

Need $\boxed{\lambda - \mu = -m^2}$

$$Y_m(y) = \sin my$$

Aha! $\lambda = \mu - m^2 = -n^2 - m^2 = -(n^2 + m^2)$.

$$\boxed{\lambda = -(n^2 + m^2)}$$

$$\frac{1}{c^2} \frac{T''}{T} = \lambda$$

$$T'' + c^2 (n^2 + m^2) T = 0$$

$$T = A \underbrace{\cos c\sqrt{n^2+m^2}\, t}_{\substack{\text{for solving} \\ u(x,y,0) = f(x,y)}} + B \underbrace{\sin c\sqrt{n^2+m^2}\, t}_{\substack{\text{for solving} \\ \frac{\partial}{\partial t} u(x,y,0) = g(x,y)}}$$

Get basic sol^{n}s $X_n Y_m T_{nm}$

Try $u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Big(A_{nm} \sin nx \sin my \cos c\sqrt{n^2+m^2}\,t$

$+ B_{nm} \sin nx \sin my \sin c\sqrt{n^2+m^2}\,t \Big)$

Last: Initial conditions:

want

$$u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin nx \sin my = f(x,y)$$

Double Fourier Series.

One way: $\varphi_{nm} = \sin nx \sin my$

are <u>orthogonal</u> $\int_0^{\pi} \int_0^{\pi} g(x,y)\,\overline{h(x,y)}\,dx\,dy$

Mult by φ_{nm} and integrate over square:

$$\int_0^{\pi} \int_0^{\pi} f(x,y) \sin nx \sin my \, dx\,dy = A_{nm} \underbrace{\int_0^{\pi} \int_0^{\pi} \sin^2 nx \sin^2 my \, dx\,dy}_{\frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}}$$

Solve for A_{nm}.

$\frac{\partial u}{\partial t}(x,y,0) =$ No A_{nm} terms, Get B_{nm} terms.

Overtones of a square drum:

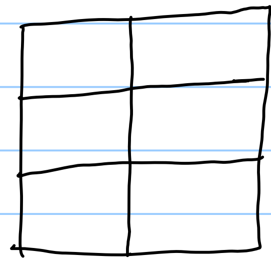
frequencies $c\sqrt{n^2+m^2}$

Discordant!

Pitch of drum: lowest one. $c\sqrt{1^2+1^2} = c\sqrt{2}$

Modes of oscillation

$\sin 2x \sin 3y$



Nodal curves

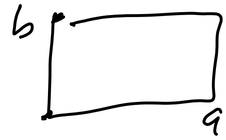
$65 = n^2+m^2$ in more than <u>two</u> ways

<u>Famous problem</u>: Can you hear the shape of a drum?

φ_{nm} are eigenfunctions for Δ.

$$\Delta \varphi_{nm} = \lambda \varphi_{nm} \qquad \lambda = \underbrace{n^2+m^2}_{\text{eigenvalue}}$$

Rectangular drum:



$$u(x,y,t) = \sum_{n,m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \cos c\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\, t$$

$$+ \sum \quad \sin$$

Prob: Among rec drums with area A, which is lowest?

$$f = c\pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$$

Ans: Square drum!

$$A = ab \qquad f = c\pi\sqrt{\frac{1}{a^2}+\frac{a^2}{A^2}}$$