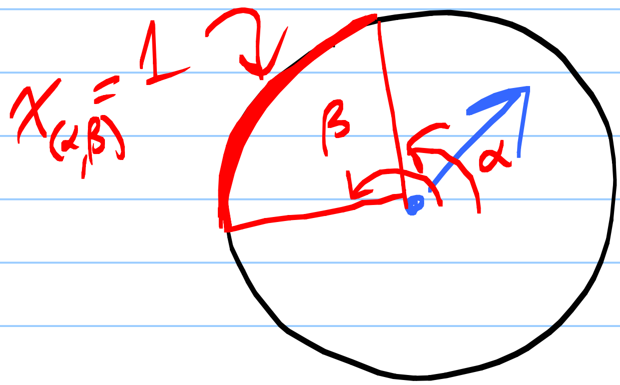
Lecture 26 Weyl's Equidistribution Thm



Spin the needle

$$\frac{\#\text{hits in 1st quad out of } N \text{ spins}}{N} \sim \frac{1}{4}$$

$$\frac{\#\text{hits in } (\alpha,\beta) \text{ out of } N}{N} \sim \frac{\beta-\alpha}{2\pi}$$

Thm: If γ is an irrational multiple of π, then $x_n = e^{in\gamma}$ are equidistributed on the unit circle, i.e.,

$$R_N = \frac{\#\text{times } e^{in\gamma} \text{ falls in } (\alpha,\beta),\ n=1,2,\dots,N}{N}$$

$$\to \frac{\beta-\alpha}{2\pi} \quad \text{as } N\to\infty.$$

$$\chi = \chi_{(\alpha,\beta)}(\theta) = \begin{cases} 1 & e^{i\theta} \in \{e^{i\psi} : \alpha < \psi < \beta\} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \theta \in (\alpha,\beta) \text{ modulo } 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$R_N = \frac{1}{N} \sum_{n=1}^{N} \chi(e^{in\gamma}) \longleftarrow \text{Hmmm}$$

$\frac{1}{2\pi} \underbrace{\frac{2\pi}{N}}_{\Delta x} \sum$ Riemann sum!

$$R_N \xrightarrow{?} \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi \, d\theta = \frac{\beta-\alpha}{2\pi}$$

<u>Lemma</u>: f continuous 2π-periodic.

$$\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

as $N \to \infty$.

<u>Pf</u>: <u>Step 1</u>: True for the Fourier series fcns:

Check for $f(\theta) \equiv 1$:

$$\frac{1}{N}\left(\sum_{n=1}^{N} 1\right) \stackrel{?}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \, d\theta \quad \checkmark$$

Check for $f(\theta) = e^{im\theta}$

$$\frac{1}{N}\left(\sum_{n=1}^{N} \underbrace{e^{imn\gamma}}_{f(n\gamma)}\right) \xrightarrow{?} \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta} \, d\theta}_{0}$$

$$\frac{1}{N} \underbrace{\sum_{n=1}^{N} \left(e^{im\gamma}\right)^{n}}_{e^{im\gamma} \sum_{n=0}^{N-1} \left(e^{im\gamma}\right)^{n}} = \frac{e^{im\gamma}}{N} \, \frac{1-e^{im\gamma N}}{1-e^{im\gamma}}$$

Key: denom <u>not</u> zero for any m!

$$\longrightarrow 0 \text{ as } N \to \infty. \checkmark$$

<u>Step 2</u>: Lemma holds for Trig Polys!

<u>Step 3</u>: f continuous 2π-periodic.

Let $\varepsilon > 0$.

Féjér's $\Longrightarrow \exists$ Trig Poly $P(\theta)$

with $|f - P| < \frac{\varepsilon}{3}$ for all θ.

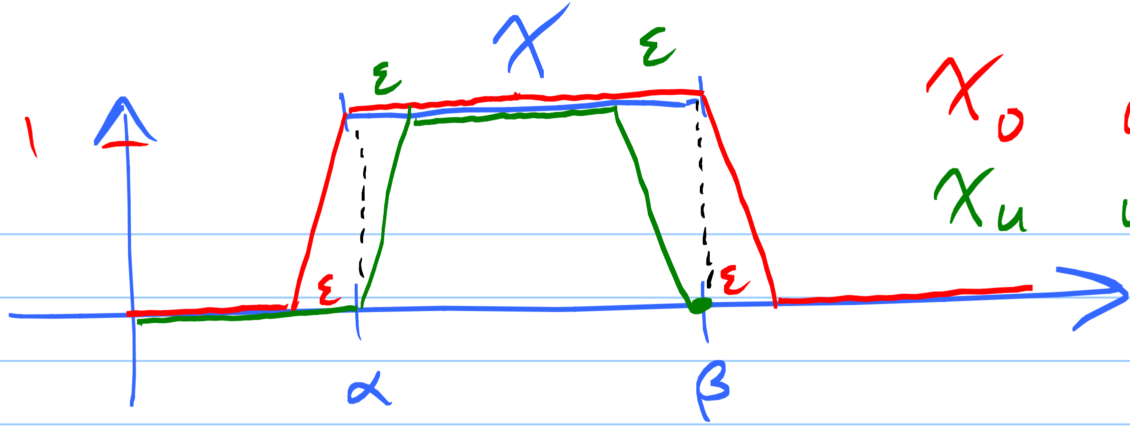
$$\left| \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta \right| =$$

$$\left| \frac{1}{N} \sum_{n=1}^{N} \big(f(n\gamma) - P(n\gamma) \big) \right. \quad \Leftarrow \quad |\sim| \le \frac{1}{N} \cdot N \frac{\varepsilon}{3} = \frac{\varepsilon}{3}$$

$$+ \frac{1}{N} \sum_{n=1}^{N} P(n\gamma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\theta)\, d\theta \quad \Leftarrow \quad |\sim| < \frac{\varepsilon}{3} \text{ if } N > N_0$$

$$\left. + \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} \big(P(\theta) - f(\theta) \big)\, d\theta}_{|\sim| \le \frac{1}{2\pi} \left(\operatorname{Max} |P - f| \right) \cdot 2\pi < \frac{\varepsilon}{3}} \right|$$

$$\le |\sim| + |\sim| + |\sim| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ \checkmark$$



χ_o o for over

χ_u u for under

$$\chi_u \le \chi \le \chi_o$$

$$\frac{1}{N}\sum_{n=1}^{N} \chi_u(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N} \chi(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N} \chi_o(e^{in\gamma})$$

Lemma $\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$$\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_u \, d\theta}_{\frac{\beta-\alpha-\varepsilon}{2\pi}} \qquad\qquad \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_o \, d\theta}_{\frac{\beta-\alpha+\varepsilon}{2\pi}}$$

Finally, let $\varepsilon \downarrow 0$ to see

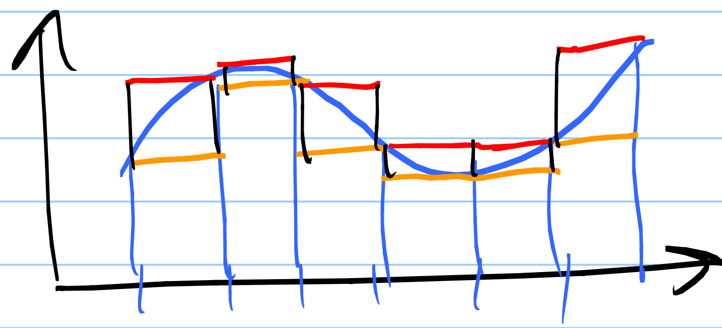
$$\frac{1}{N}\sum_{n=1}^{N} \chi(e^{in\gamma}) \to \frac{\beta-\alpha}{2\pi}$$

✓

Done!

Hmmm. Lemma holds for step fcns.

Continue same reasoning to see the Lemma holds for Riemann Integrable fcns.



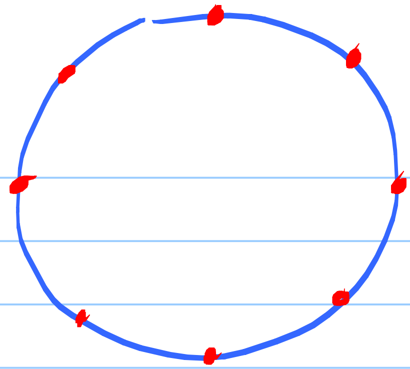
Herman Weyl (Vile)

André Weil (Vay)

Weyl's criterion: $e^{i\theta_n}$ is equidistributed in unit circle $\iff$ $\frac{1}{N}\sum_{n=1}^{N} e^{im\theta_n} \to 0$

as $N \to \infty$ for each m.

What if γ is rational multiple of π?



$\frac{1}{4}\pi$

Billiard problem: