Lecture 28 Fourier Integral

Another angle on Ch. 3: Ex 20:

$$S_N(x) = \sum_{n=1}^{N} \frac{\sin nx}{n}$$

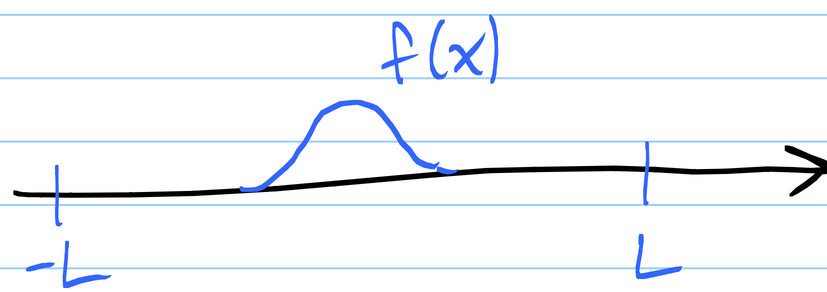
$$S_N\left(\frac{\pi}{N}\right) = \underbrace{\left(\sum_{n=1}^{N} \frac{\sin n \cdot \frac{\pi}{N}}{n \cdot \frac{\pi}{N}}\right) \underbrace{\frac{\pi}{N}}_{\Delta x}}_{\text{Riemann sum!}}$$

$$\longrightarrow \int_0^{\pi} \frac{\sin t}{t}\, dt$$

Fourier Series: 2π periodic $f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

Idea:



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

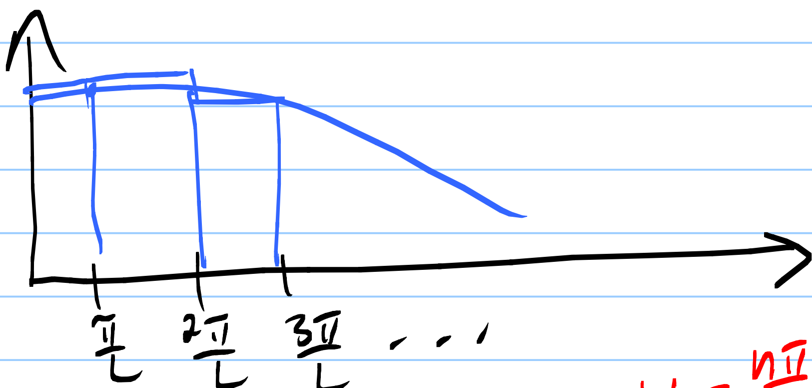
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos nt\,dt$$

$$b_n = \; \cdots\cdots \; \sin nt \; \cdots\cdots$$

$n\frac{\pi}{L}$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\overset{\pi}{\cancel{1}}}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt\right)\cos\frac{n\pi x}{L}$$

$$+ \sum \cdots \sin \cdots \sin$$



Let $L \to \infty$

$w_n = \frac{n\pi}{L}$

Riemann Sum!

$$f(x) = \lim_{L\to\infty}\sum_{1}^{\infty}\frac{1}{\pi}\left(\int_{-\infty}^{\infty} f(t)\cos w_n t\,dt\right)\cos w_n x \underbrace{\left(\frac{\pi}{L}\right)}_{\Delta w}$$

$$+ \cdots \sin \cdots \sin t$$

$$\boxed{f(x) = \int_0^{\infty} A(w)\cos wx\,dt + \int_0^{\infty} B(w)\sin wx\,dt}$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$

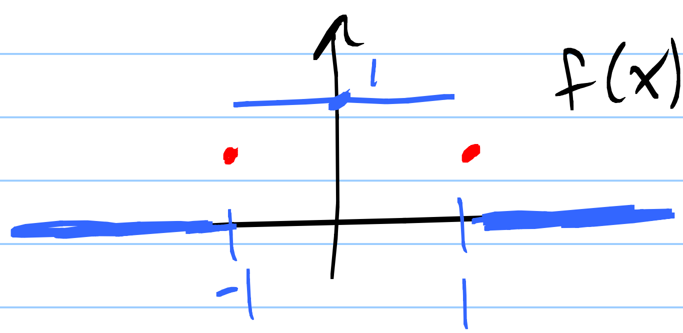
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

Fact: If f is piecewise C^1-smooth and $\int_{-\infty}^{\infty} |f|\,dx$ is finite, then f is "given by" its Fourier Integral:

= where f is continuous

= midpoint of jumps.

EX:



$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

even · even = even

$$= \frac{2}{\pi} \int_0^\infty f(t) \cos wt \, dt$$

$$= \frac{2}{\pi} \int_0^1 \cos wt \, dt$$

$$= \frac{2}{\pi} \left[\frac{1}{w} \sin wt \right]_0^1 = \frac{2}{\pi} \frac{\sin w}{w}$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\underset{\text{even}}{f(t)} \; \underset{\text{odd}}{\sin wt}}_{\text{odd}} \, dt = 0$$

$$f(x) \text{ “=” } \int_0^\infty \underbrace{\frac{2}{\pi} \frac{\sin w}{w}}_{A(w)} \cos wx \, dw$$

$$= \begin{cases} 0 & |x| > 1 \\ 1 & |x| < 1 \\ 1/2 & x = \pm 1 \end{cases}$$

Aha! $x=0$:

$$f(0) = 1 = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\, dw \quad !$$

<u>Fact</u>: f even: $B(w) \equiv 0$

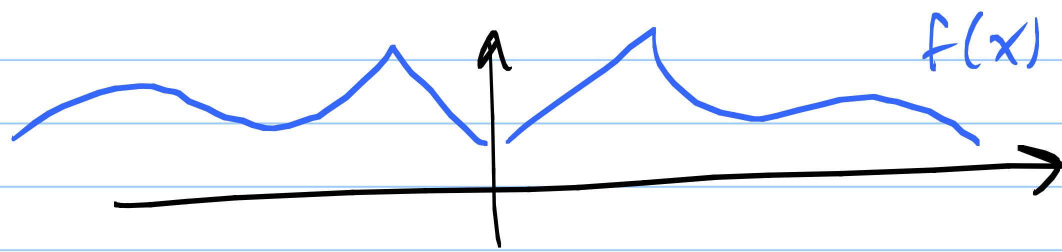
$$A(w) = \frac{2}{\pi}\int_0^\infty f(t)\cos wt\, dt$$

$$f(x) = \int_0^\infty A(w)\cos wx\, dw$$

Fourier Cosine Transform: $f(x)$, $x \geq 0$

$$\tilde{\mathcal{F}}_c[f] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos wt\, dt$$

Aha! $\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c = \text{Identity}.$



extend as a even fcn.

Fact: f odd. $A(w) \equiv 0$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt$$

$$f(x) = \int_0^\infty B(w)\sin wx\,dw$$

Fourier Sine Transform:

$$\tilde{\mathcal{F}}_s[f](w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin wt\,dt$$

$\tilde{\mathcal{F}}_s \circ \tilde{\mathcal{F}}_s$ = back to f.

Properties: 1) $\tilde{\mathcal{F}}_c[f'] = w\,\tilde{\mathcal{F}}_s[f] - \sqrt{\frac{2}{\pi}}f(0)$

2) $\tilde{\mathcal{F}}_s[f'] = -w\,\tilde{\mathcal{F}}_c[f]$

Why 1: $\tilde{\mathcal{F}}_c[f'] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(t) \underbrace{\cos wt}_{u}\, dt$

$dv = f'(t)\,dt$

$v = f(t)$

$$= \lim_{B\to\infty} \sqrt{\frac{2}{\pi}} \int_0^B \text{-----}\, dt$$

$$= \lim_{B\to\infty} \sqrt{\frac{2}{\pi}} \left[f(t)\cos wt \Big|_0^B - \int_0^B f(t)[-w\sin wt]\,dt \right]$$

need $f(t) \to 0$ as $t \to \infty$

$$= -\sqrt{\frac{2}{\pi}} f(0) + \underbrace{w\sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin wt\, dt}_{\tilde{\mathcal{F}}_s[f]} \checkmark$$

Consequence: Do f' formulas twice:

a) $\tilde{\mathcal{F}}_c[f''] = -w^2 \tilde{\mathcal{F}}_c[f] - \sqrt{\frac{2}{\pi}} f'(0)$

$\tilde{\mathcal{F}}_s[f''] = -w^2 \tilde{\mathcal{F}}_s[f] + \sqrt{\frac{2}{\pi}} f(0)$