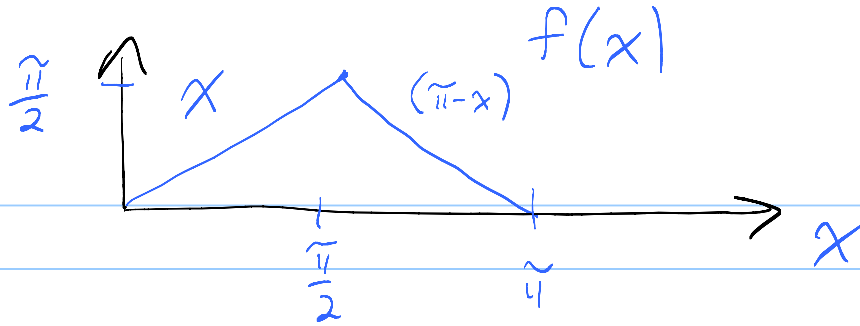
Lecture 3



$$\left[f(x) \stackrel{?}{=} \sum_{n=1}^{\infty} A_n \sin nx \right] \cdot \sin mx$$

$$\int_0^{\pi} f(x) \sin mx \, dx = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{=0 \text{ if } n \neq m}$$

$$= A_m \underbrace{\int_0^{\pi} \sin^2 mx \, dx}_{= \pi/2}$$

Coeff determined: $\boxed{A_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx \, dx}$

$$A_n = \frac{2}{\pi} \left[\int_0^{\pi/2} \underset{u}{x} \cdot \underbrace{\sin nx \, dx}_{dv} + \int_{\pi/2}^{\pi} \underbrace{(\pi - x)}_{u} \underbrace{\sin nx \, dx}_{dv} \right]$$

$$\int u \, dv = uv - \int v \, du$$

$A_1 = \frac{4}{\pi} \cdot \frac{1}{1^2}$ $\qquad A_2 = A_4 = A_6 = \cdots \quad 0$

$A_3 = \frac{4}{\pi}\left(\frac{-1}{3^2}\right), \quad A_5 = \frac{4}{\pi}\left(\frac{1}{5^2}\right), \quad \ldots$

$$f(x) \stackrel{?}{=} \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right)$$

Know $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\frac{\pi^2}{6}} < \infty$ Integral test.

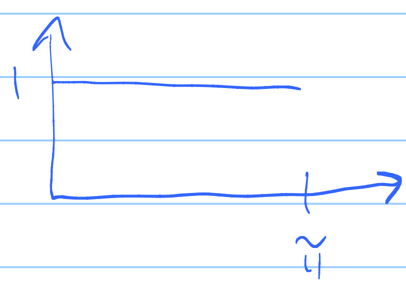
Compare to $\int_1^\infty \frac{1}{x^2}\,dx < \infty$

$$\left|\frac{1}{n^2}\sin nx\right| \leq \frac{1}{n^2}$$

Weierstraß M-test shows this series converges uniformly on $[0, \pi]$.

Fact: Uniform limit of continuous fcns is a cont. fcn.

EX:



$f(x) = 1$

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

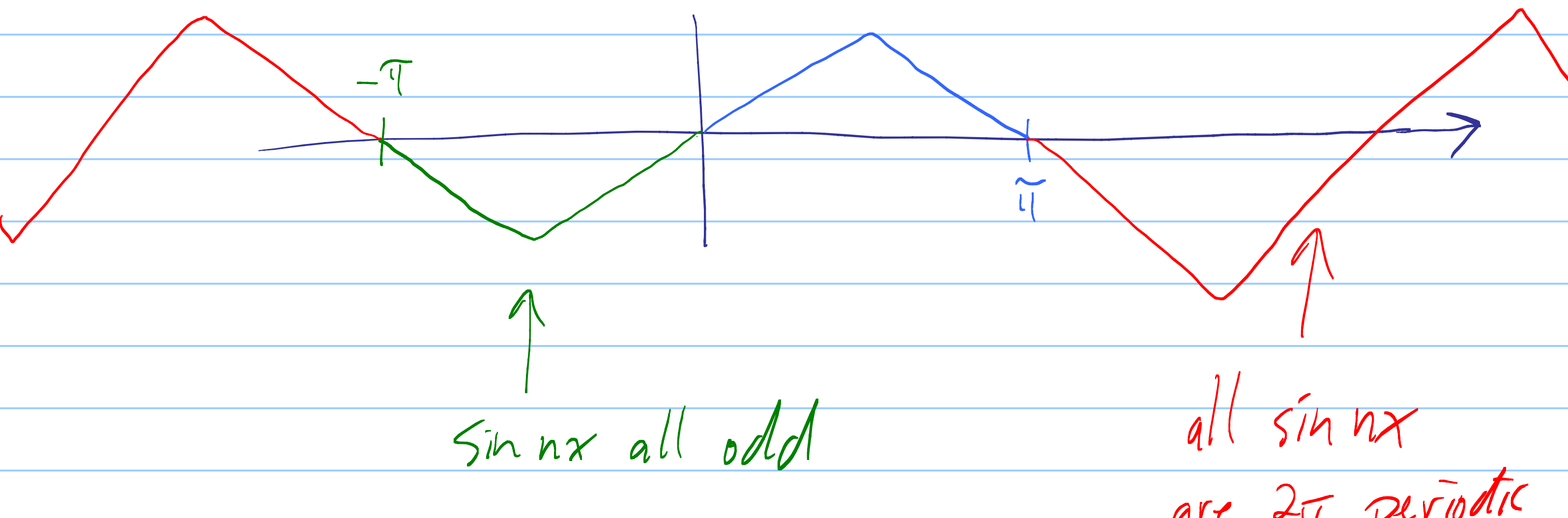
$$= \frac{2}{\pi}\int_0^\pi \sin nx\,dx$$

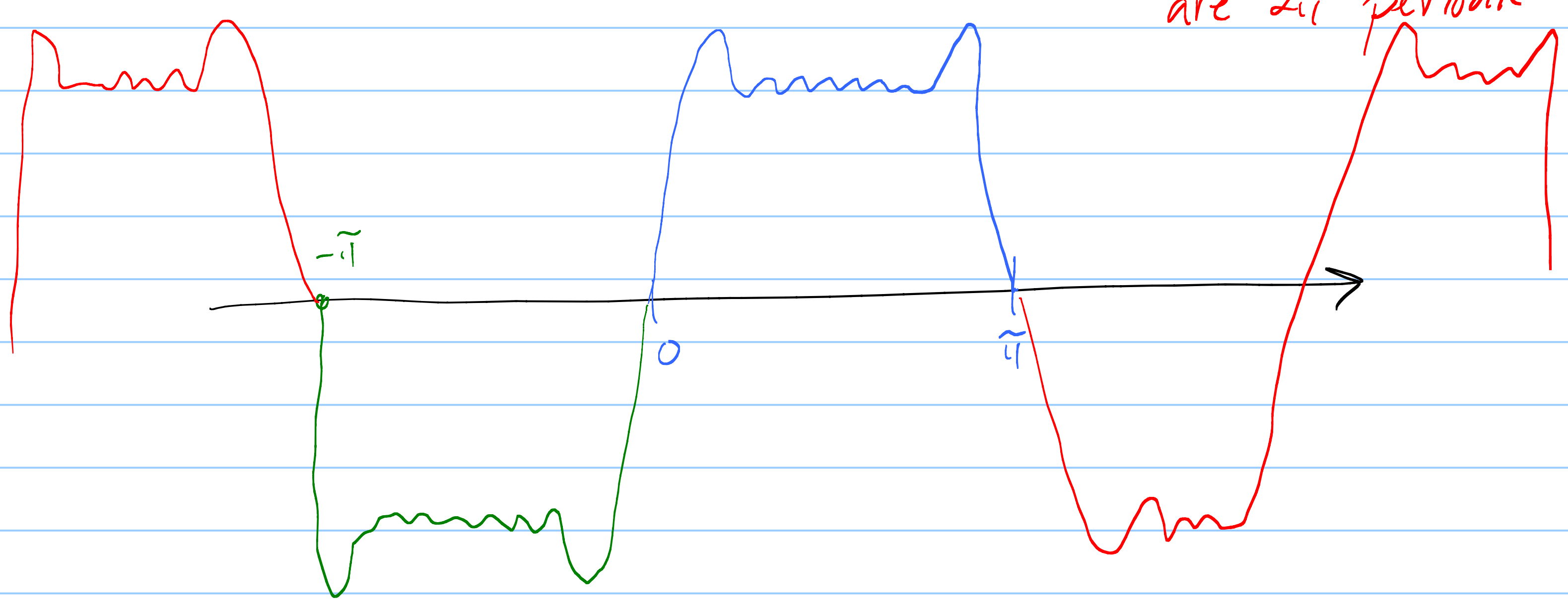
$$= \frac{2}{\pi} \left[-\frac{1}{n} \cos nx \right]_0^{\pi}$$

$$= \frac{2}{\pi n} \left[1 - \cos n\pi \right] = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd} \end{cases}$$

$$f(x) \stackrel{?}{=} \frac{4}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right]$$

<u>Extensions of the series to $\mathbb{R}$:</u>





Full Fourier Series. $[-\pi, \pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \sin nx + b_n \cos nx \right)$$

Big fact: 1, $\sin nx$, $\cos nx$ are orthogonal fcns on $[-\pi, \pi]$.

Check:

$$\int_{-\pi}^{\pi} 1 \cdot \sin nx \, dx = 0$$

$$\int_{-\pi}^{\pi} 1 \cdot \cos nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad n \neq m$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad n \neq m$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0$$

Can see that if $f(x) = $ F. S., then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \text{Ave of } f \text{ on } [-\pi, \pi]$$

$$\int_{-\pi}^{\pi} 1 \cdot f(x)\,dx = \int \left[a_0 + \left(\sum \text{trig} \right) \right] \cdot 1 \, dx$$

$$= 2\pi a_0 + \left(\sum 0\text{'s} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$\pi = \int_{-\pi}^{\pi} \cos^2 nx \, dx$$

$$\text{Fourier Series} = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$