# Lecture 30

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} \quad \text{where}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$$

Repeat Riemann Integral trick.
Assume $f$ supported on a small set.
$[-L, L]$ and let $L \to \infty$.

$$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{2\pi i x s}\, ds$$

$$\text{where} \quad F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t s}\, dt$$

## Notation:

$$\mathcal{F}[f] = \hat{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i s t}\, dt$$

is the Fourier Transform.

The inverse F. T. is

$$\mathcal{F}^{-1}[g] = \check{g}(x) = \int_{-\infty}^{\infty} g(s)\, e^{2\pi i x s}\, ds$$

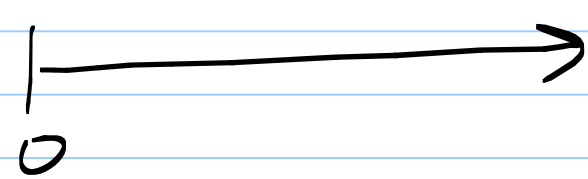
Big Fact: $\mathcal{F}^{-1} \circ \mathcal{F} = \text{identity} \quad \left( \check{\hat{f}} = f \right)$

Other properties:

$$\mathcal{F}[f'] = 2\pi i s \hat{f}(s)$$

$$\mathcal{F}[f''] = -4\pi^2 s^2 \hat{f}(s)$$

Heat problems:




Left end held at 0 degrees:

Use Fourier Sine Transform:

$$\mathcal{F}_s[u_{xx}] = -w^2 \mathcal{F}_s[u] + \sqrt{\frac{2}{\pi}}\, u(0,t)$$

$= 0$

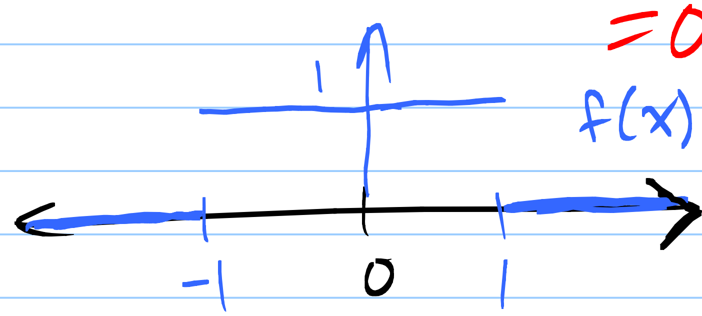
Left end insulated: $\frac{\partial}{\partial x} u(0,t) = 0$

temp gradient

Use Fourier Cosine Transform:

$$\tilde{\mathcal{F}}_c[u_{xx}] = -w^2 \tilde{\mathcal{F}}_c[u] - \sqrt{\frac{2}{\pi}} \underbrace{\frac{\partial u}{\partial x}(0,t)}_{=0}$$

No end problem:




$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{Apply F.T. in space var.}$$

$$\hat{u}(s,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-2\pi i x s}\, dx$$

$$\frac{\partial \hat{u}}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\, e^{-2\pi i x s}\, dx$$

because heat $\to 0$ "rapidly"

at $\infty$."

$$\tilde{\mathcal{F}}[u_{xx}] = \tilde{\mathcal{F}}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial \hat{u}}{\partial t}$$

$$\parallel$$

$$-4\pi^2 s^2 \hat{u}(s,t)$$

$$\boxed{\frac{\partial \hat{u}}{\partial t} = -4\pi^2 s^2 \hat{u}}$$

$$\hat{u} = C(s)\, e^{-4\pi^2 s^2 t}$$

Use initial condition to pin down $C(s)$:

$$\hat{u}(s,0) = C(s) = \int_{-\infty}^{\infty} \underbrace{u(x,0)}_{f(x)}\, e^{-2\pi i s x}\, dx$$

$$= \int_{-1}^{1} 1 \cdot e^{-2\pi i s x}\, dx$$

$$= \left[\frac{-1}{2\pi i s} e^{-2\pi i s x}\right]_{x=-1}^{1}$$

$$= \frac{1 \cdot 2i}{2\pi i s}\left(\frac{e^{2\pi i s} - e^{-2\pi i s}}{2i}\right)$$

$$= \frac{1}{\pi} \cdot \frac{\sin 2\pi s}{s} = C(s)$$

Now $\hat{u}(s,t) = \frac{1}{\pi} \frac{\sin 2\pi s}{s} e^{-4\pi^2 s^2 t}$

Undo $\mathcal{F}$ with $\mathcal{F}^{-1}$:

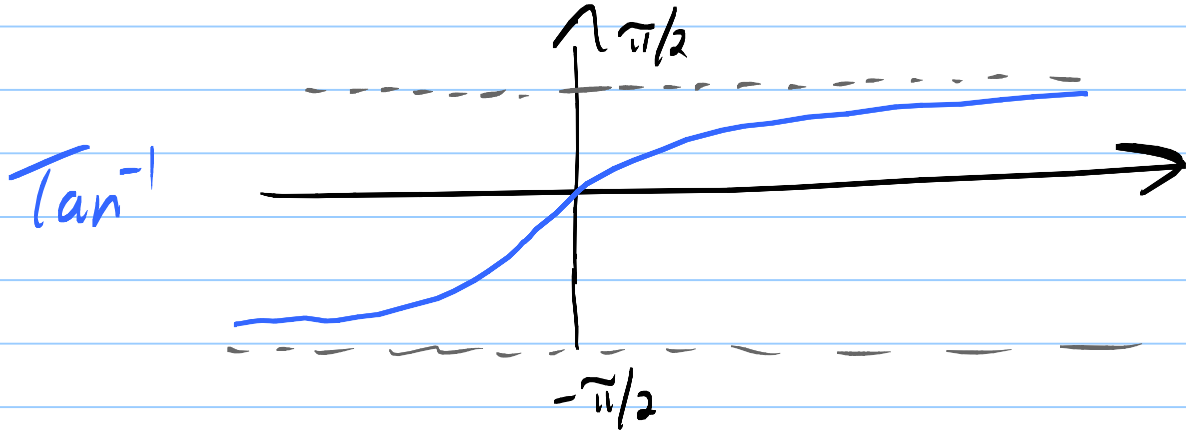
$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\pi} \underbrace{\frac{\sin 2\pi s}{s}}_{\text{even}} \underbrace{e^{-4\pi^2 s^2 t}}_{\text{even}} \underbrace{e^{2\pi i s x}}_{\underbrace{\cos 2\pi s x}_{\text{even}} + i \underbrace{\sin 2\pi s x}_{\text{odd}}} ds$$

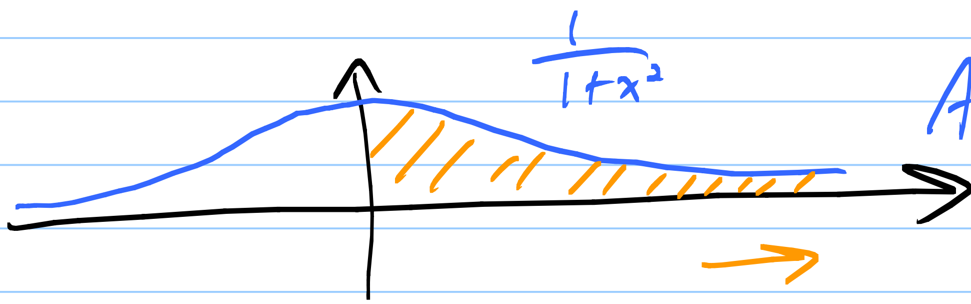
$$\boxed{u = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin 2\pi s}{s} e^{-4\pi^2 s^2 t} \cos 2\pi s x \, ds}$$

Read Chapt. 5.

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{B\to\infty} \int_0^B \frac{1}{1+x^2}\,dx$$

$$= \lim_{B\to\infty} \text{ArcTan}\, x \Big]_0^B = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$




Area $= \frac{\pi}{2}$

$$\int_{-\infty}^{\infty} = \lim_{A\to -\infty} \int_A^0 + \lim_{B\to\infty} \int_0^B$$

$$= \lim_{N\to\infty} \int_{-N}^{N} \quad \text{when everything is "nice"}$$