

Lecture 31 Bump fns

$\mathcal{M}(\mathbb{R})$ = "fns on $\mathbb{R}$ of moderate decrease at infinity"

$$= \left\{ f : |f(x)| \leq \frac{M}{1+x^2} \text{ for all } x, \right. \\ \left. M > 0. \right\}$$

Since $\int_0^\infty \frac{1}{1+x^2} = \frac{\pi}{2} < \infty$, fns in $\mathcal{M}$ are absolutely integrable.

$$\text{So } \int_{-\infty}^{\infty} f \, dx = \int_{-\infty}^a f \, dx + \int_a^{\infty} f \, dx$$

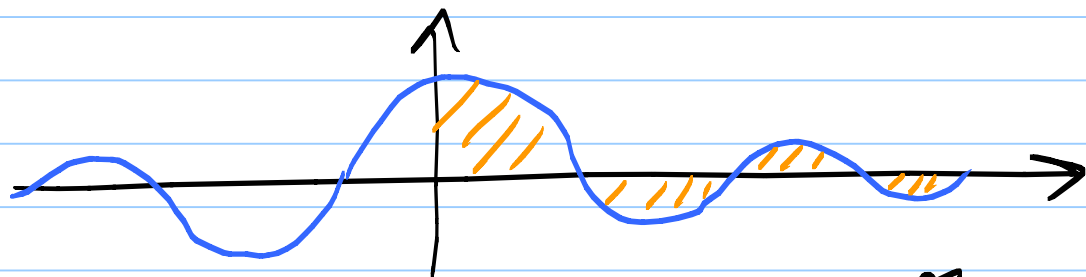
$$= \lim_{A \rightarrow -\infty} \int_A^a f \, dx + \lim_{B \rightarrow \infty} \int_a^B f \, dx$$

both limits exist

if $f \in \mathcal{M}$.

And $\int_{-\infty}^{\infty} f \, dx = \lim_{N \rightarrow \infty} \int_{-N}^N f \, dx.$

Note: $\frac{\sin x}{x}$ is not in $\mathcal{M}$.



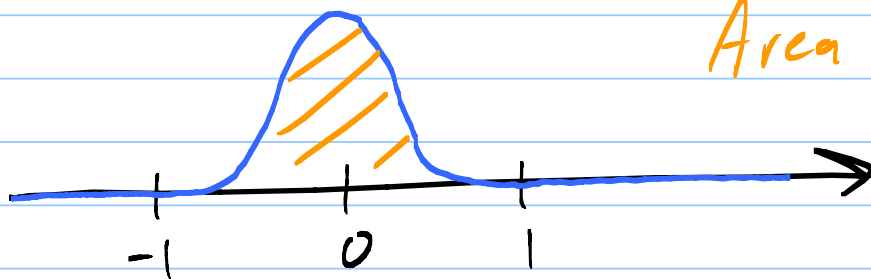
Conditionally convergent $\int_0^{\infty}$, not absolutely integrable.

Integration facts for fcn's in $\mathcal{M}$:

$$\int_{-\infty}^{\infty} f(x+h) \, dx = \int_{-\infty}^{\infty} f(x) \, dx$$

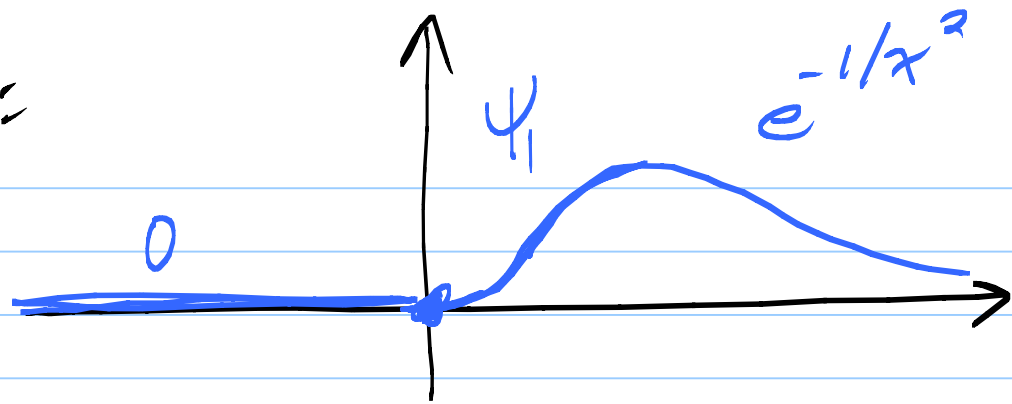
see Stein.

Bump fcn in $C_0^{\infty}(\mathbb{R})$:



Area = 1

Step 1:



Clear that ψ_1 is continuous at $x=0$

$$\psi_1' = e^{-1/x^2} \left[\frac{2}{x^3} \right]$$

$$= 2e^{-u} \cdot u^{3/2} \text{ where } u = \frac{1}{x^2} \rightarrow \infty \text{ as } x \rightarrow 0$$

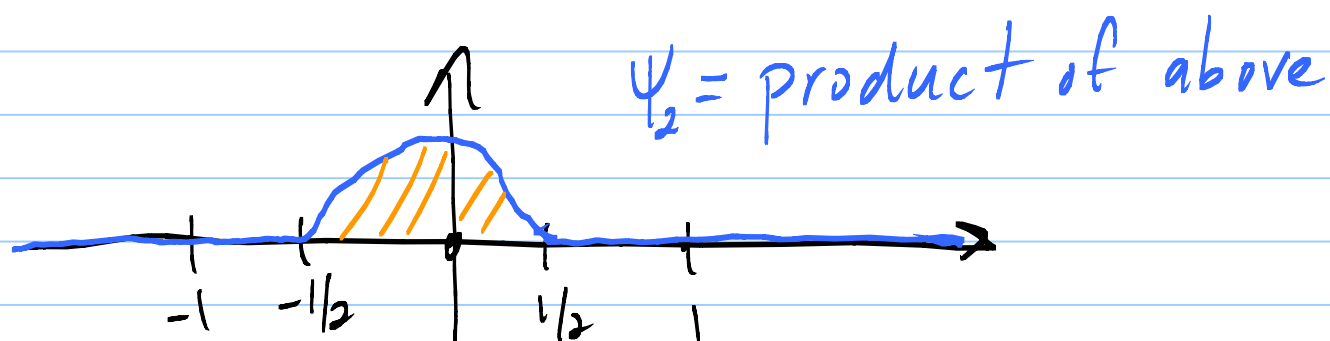
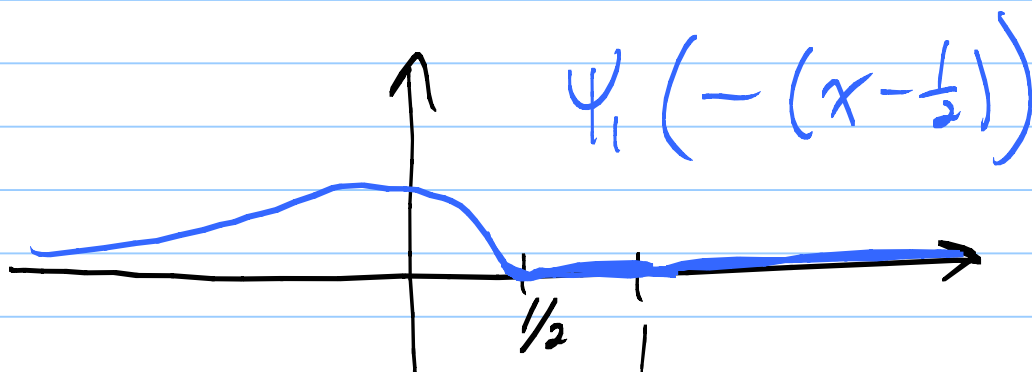
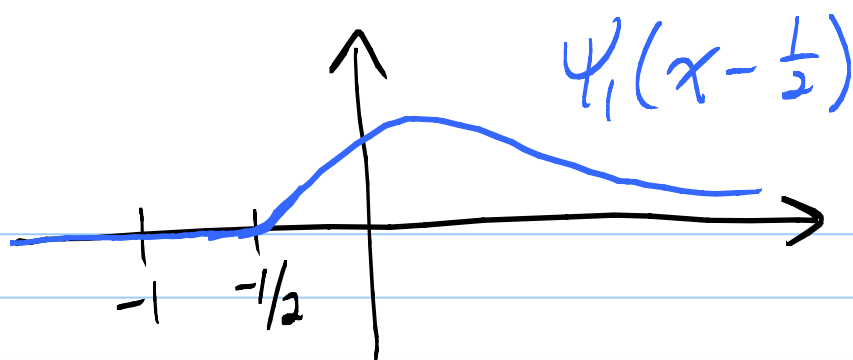
e^{-u} dies away so fast, we see $\psi_1' \rightarrow 0$ as $x \rightarrow 0$.

etc. See all derivatives of ψ_1 exist at 0.

So ψ_1 is C^∞ smooth on $\mathbb{R}$.

Fact: ψ_1 is not equal to its power series near $x=0$. [power series $\equiv 0$, but $\psi_1 \not\equiv 0$ near 0.]

Step 2: To get a "bump"



$$\text{Let } c = \int_{-\infty}^{\infty} \psi_2 dx = \int_{-1/2}^{1/2} \psi_2 dx > 0$$

$$\varphi = \frac{1}{c} \psi_2.$$

Properties: $\varphi \in C_0^\infty(\mathbb{R})$

$$1) \text{ supp } \varphi = \{x : \varphi(x) \neq 0\} \subset [-\frac{1}{2}, \frac{1}{2}]$$

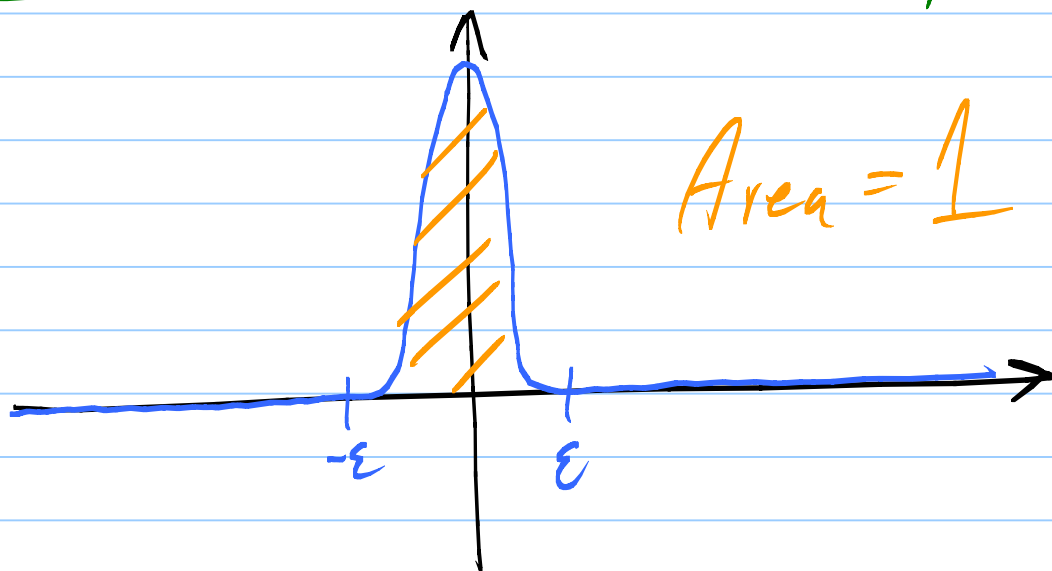
" φ has compact support"

2) Symmetric $\varphi(-x) = \varphi(x)$

3) $\int_{-\infty}^{\infty} \varphi \, dx = 1$

4) $\varphi(x) \geq 0$

Approximation to the identity:



$$\varphi_\epsilon(x) = \frac{1}{\epsilon} \varphi\left(\frac{x}{\epsilon}\right)$$

$$\int_{-\infty}^{\infty} \varphi_\epsilon(x) \, dx = \int_{-\epsilon}^{\epsilon} \varphi_\epsilon(x) \, dx$$

$$= \int_{-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right) dx = \int_{-1}^1 \varphi(u) du = 1$$

$\uparrow$
 $u = \frac{x}{\varepsilon}$

$du = \frac{1}{\varepsilon} dx$

When $x = -\varepsilon$: $u = \frac{-\varepsilon}{\varepsilon} = -1$
 $x = \varepsilon$: $u = \frac{\varepsilon}{\varepsilon} = 1$

"Approx to identity": $\varphi_\varepsilon * f \approx f$

Convolution: $f * g(x) = \int_{-\infty}^{\infty} f(x-t) g(t) dt$

Well defined when $f, g \in \mathcal{M}$.

Think about $(\varphi_\varepsilon * f)(x)$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(\underbrace{x-t}_{\tilde{\tau}}) f(t) dt$$

$$d\tilde{\tau} = -dt$$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(\tilde{\tau}) f(x-\tilde{\tau}) d\tilde{\tau}$$

"Good kernel trick"

$$\begin{aligned} f(x) &= f(x) \cdot 1 = f(x) \int_{-\infty}^{\infty} \varphi_{\varepsilon}(\tau) d\tau \\ &= \int_{-\infty}^{\infty} \varphi_{\varepsilon}(\tau) f(x) d\tau \end{aligned}$$

$$(\varphi_{\varepsilon} * f)(x) - f(x) =$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi_{\varepsilon}(\tau) [f(x-\tau) - f(x)] d\tau \\ &= \int_{-\varepsilon}^{\varepsilon} \varphi_{\varepsilon}(\tau) \underbrace{[f(x-\tau) - f(x)]}_{|f(x-\tau) - f(x)| < \varepsilon \text{ if } |\tau| < \delta} d\tau \end{aligned}$$

$$\begin{aligned} \text{Now } |(\varphi_{\varepsilon} * f) - f| &\leq \left| \int_{-\varepsilon}^{\varepsilon} \right| \leq \int_{-\varepsilon}^{\varepsilon} \varphi_{\varepsilon}(\tau) \varepsilon d\tau \\ &= \varepsilon \cdot \underbrace{\int_{-\varepsilon}^{\varepsilon} \varphi_{\varepsilon}(\tau) d\tau}_1 \end{aligned}$$

if $\varepsilon < \delta$.

Fact: f continuous on $[a, b]$. Given $\varepsilon > 0$.

Then $\exists C^\infty$ fcn f_ε such that

$$|f - f_\varepsilon| < \varepsilon \text{ on } [a, b].$$

How: $f_\varepsilon = \varphi_\varepsilon * f$ ↖ extend f to $\mathbb{R}$.

$$\varphi * f = \int_{-\infty}^{\infty} \varphi(x-t) f(t) dt$$

$$DQ = \int_{-\infty}^{\infty} \frac{\varphi((x+h)-t) - \varphi(x-t)}{h} f(t) dt$$

$$= \int_A^B \text{same thing} \rightarrow \int_{-\infty}^{\infty} \varphi'(x-t) f(t) dt$$

Repeat. See $\varphi_\varepsilon * f$ is C^∞ smooth.

Weierstraß Thm: Can approximate continuous fcn on $[a, b]$ by polys.

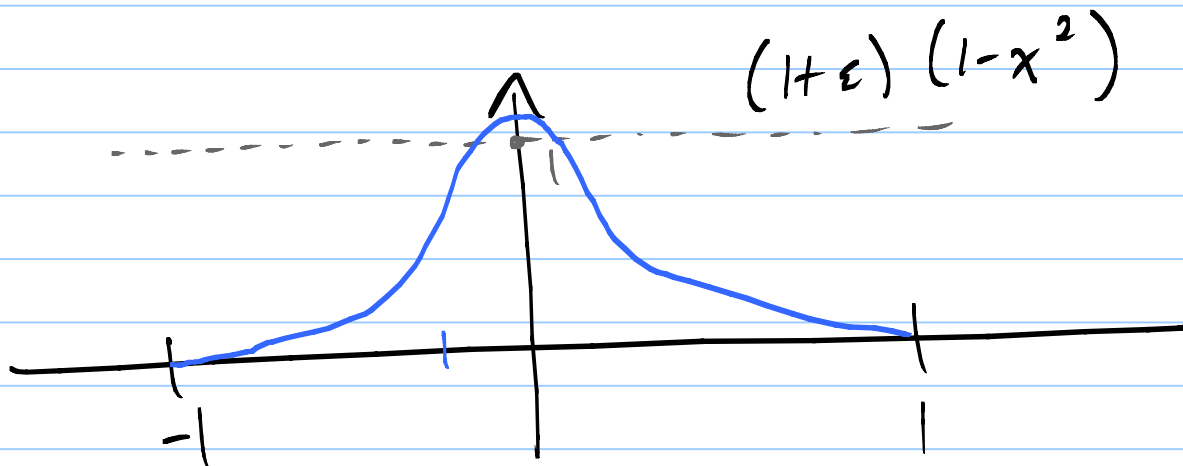
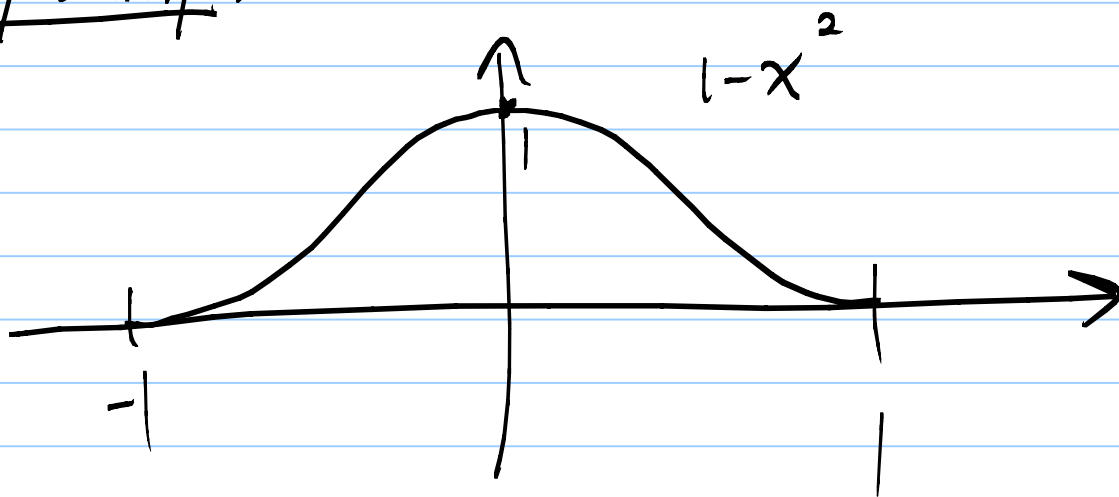
Easy, with a polynomial "bump fn".

Case: $[a, b] = [-1, 1]$. f cont on $[-1, 1]$.

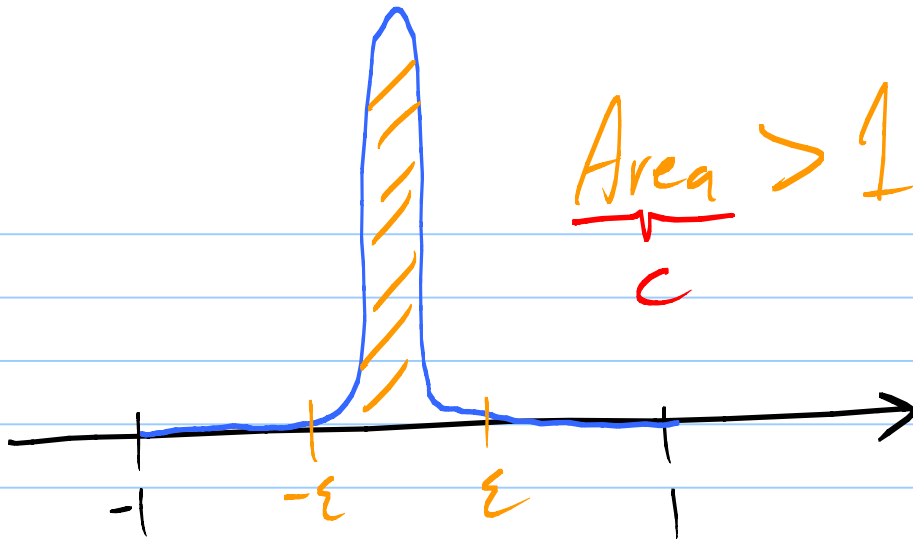
subtract off $Ax + B$ so that

$f(-1) = 0 = f(1)$. Extend f by zero to $\mathbb{R}$.

Poly bump:



Take big powers until



Multiply by $\frac{1}{c}$: $\varphi_\varepsilon(x) = \begin{cases} \frac{1}{c} [(1+\varepsilon)(1-x^2)]^N \\ 0 \text{ outside } [-1, 1]. \end{cases}$

$\varphi_\varepsilon * f$ is close to f .
 φ_ε $\nwarrow$ polynomial!