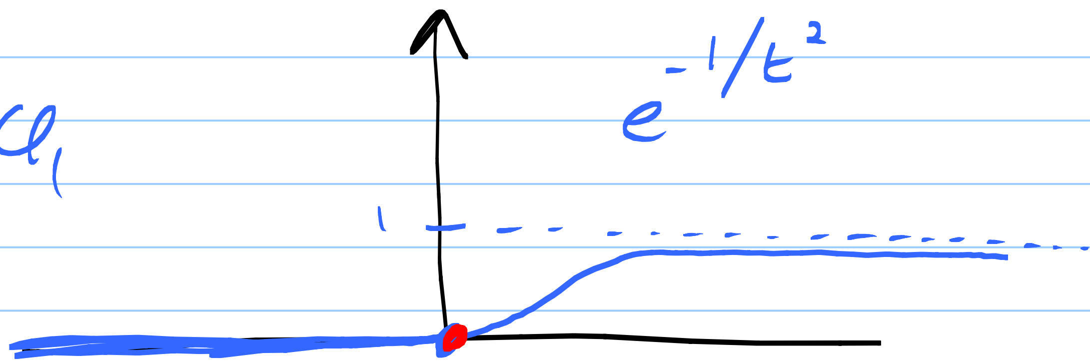
Lecture 32

φ_1



φ_1 cont at 0 ✓

φ_1 diff'ble at 0:

$$\frac{\varphi_1(0+h)-\varphi_1(0)}{h} = \frac{e^{-1/h^2}-0}{h}$$

$$= \frac{e^{-1/h^2}}{h}$$

$\frac{1}{h} = t$

$t \to \infty$ as $h \to 0$.

$$= \frac{e^{-t^2}}{\frac{1}{t}} = \frac{t}{e^{t^2}} \longrightarrow 0 \quad \text{as } t \to \infty.$$

L.H. Der $= 0 =$ R.H. Der.

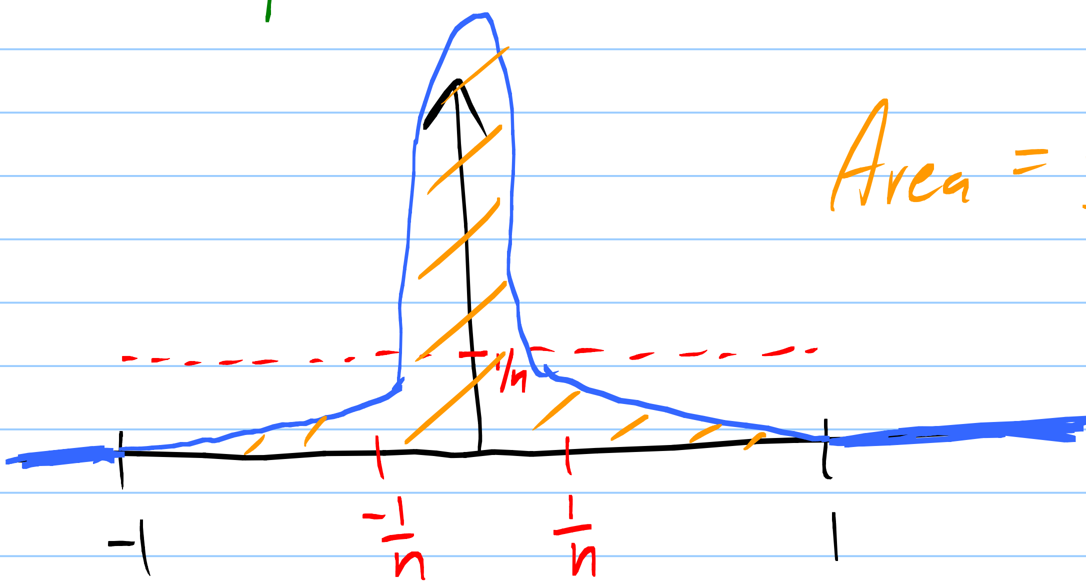
So φ_1 is diff'ble at 0. $\varphi_1'(0) = 0$.

Repeat for higher order derivatives.

Weierstraß Thm: Key: Polyomial bump fcn:

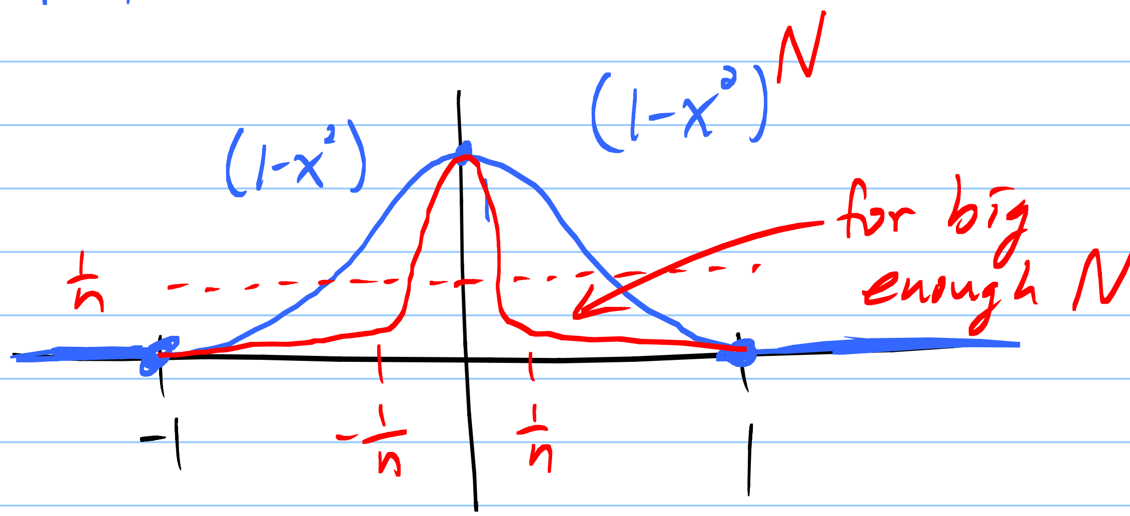
Want bump fcn such that

Q_n



Q_n = poly between -1 and 1.

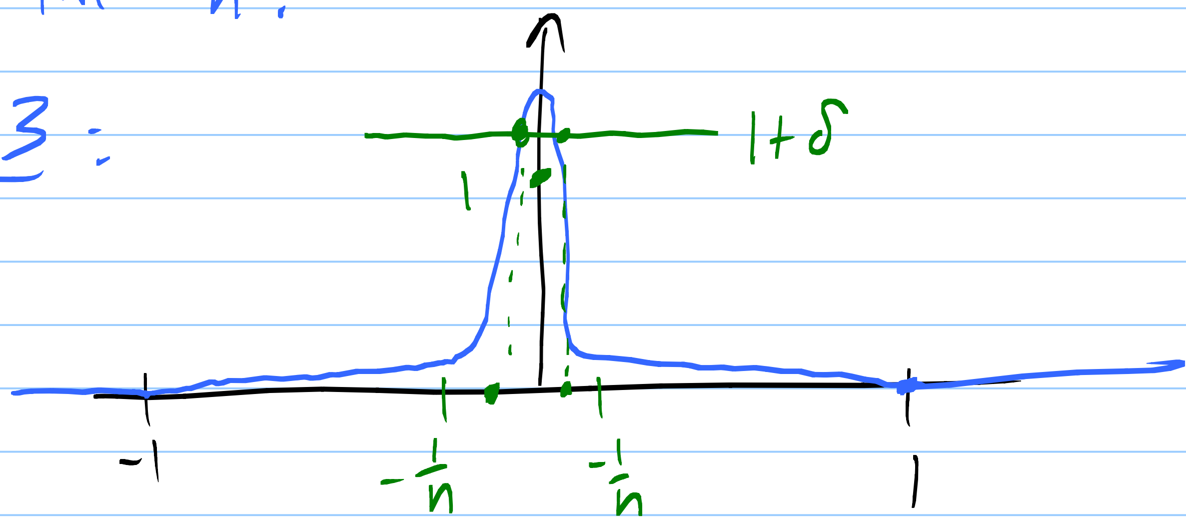
Step 1:



Step 2: $(1+\varepsilon)(1-x^2)^N$ where $\varepsilon > 0$ is

small enough that $(1+\varepsilon)(1-x^2)^N < \frac{1}{n}$
if $|x| > \frac{1}{n}$.

<u>Step 3</u>:



Now take $\left[(1+\varepsilon)(1-x^2)^N\right]^M$

with M so big that area > 1.

Keep good estimate for $|x| > \frac{1}{n}$.

<u>Step 4</u>: Divide by area > 1.

<u>Pf</u>: Assume $f(-1) = 0 = f(1)$.

[can subtract $ax+b$ to make this happen.] Extend f to be zero outside $[-1,1]$.

Claim: $Q_n * f$ is a poly.

$$(Q_n * f)(x) = \int_{-\infty}^{\infty} \underbrace{Q_n(x-t)}_{\substack{\sum a_m (x-t)^m \\ = \sum c_{ij} x^i t^j}} f(t)\,dt$$

$$= \int_{-1}^{1} Q_n(x-t) f(t)\,dt$$

$$= \sum c_{ij} x^i \int_{-1}^{1} t^j f(t)\,dt$$

$= \text{poly in } x.$ ✓

Claim: $Q_n * f \to f$ unif on $[-1,1]$.

$$(Q_n * f)(x) - f(x) = \int_{-\infty}^{\infty} Q_n(\underbrace{x-t}_{\tau}) f(t)\,dt - f(x) \underbrace{\int_{-\infty}^{\infty} Q(t)\,dt}_{1}$$

$$= \int_{-\infty}^{\infty} Q_n(\tau)\, f(x-\tau)\, d\tau$$

$$- \int_{-\infty}^{\infty} Q_n(\tau)\, f(x)\, d\tau$$

$$I = \int_{-\infty}^{\infty} Q_n(\tau) \left[f(x-\tau) - f(x) \right] d\tau$$

f cont. on $[-1,1] \Rightarrow f$ unif cont

on $[-1,1]$, and in fact on $\mathbb{R}$.

Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\left| f(x-\tau) - f(x) \right| < \frac{\varepsilon}{2} \quad \text{if} \quad \tau < \delta.$$

If n is such that $\frac{1}{n} < \delta$, then

$$I = \int_{|\tau| \geq \frac{1}{n}} Q_n(\tau) \left[f(x-\tau) - f(x) \right] d\tau + \int_{|\tau| < \frac{1}{n}} Q_n(\tau) \Big[\underbrace{f(x-\tau) - f(x)}_{|\cdots| < \frac{\varepsilon}{2}} \Big] d\tau$$

$< \frac{1}{n}$

f cont on closed interval $\Rightarrow |f| < M$ on $[-1,1]$, and hence on $\mathbb{R}$.

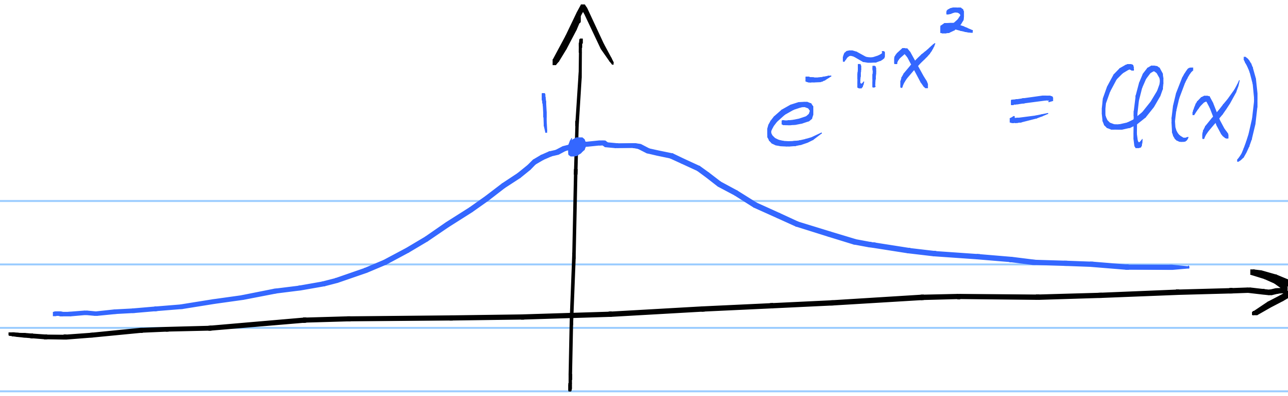
$$\underbrace{|I|}_{\text{error}} \le \frac{1}{n}(M+M)\cdot 2 + \underbrace{\int_{|\tilde{t}|<\frac{1}{n}} Q_n(\tilde{t})\,\frac{\varepsilon}{2}\,d\tilde{t}}_{< \frac{\varepsilon}{2}\underbrace{\int_{-1}^{1} Q_n(\tilde{t})\,d\tilde{t}}_{=1}}$$

$$|I| \le \frac{4M}{n} + \frac{\varepsilon}{2} \quad \text{if} \quad \frac{1}{n} < \delta.$$

Make $n > \frac{1}{\delta}$ even bigger so that

$$\frac{4M}{n} < \frac{\varepsilon}{2}.$$

<u>Important bump</u>: The Gaussian or the "bell curve".



1) $\varphi(x) > 0$. ✓

2) $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$

3) $\hat{\varphi} = \varphi$ (Whoa!)

2):
$$\left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right)^2$$

$$\left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\pi y^2}\,dy\right)$$

$$=$$

Fubinni

$$\int_{y=-\infty}^{\infty} \left(\int_{x=-\infty}^{\infty} e^{-\pi x^2} \; e^{-\pi y^2} \, dx \right) dy$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi (x^2+y^2)} \underbrace{dx\,dy}_{dA}$$

now switch to polar coords:

$$dA = r\,dr\,d\theta$$

$$\int_{\theta=0}^{2\pi} \left(\int_{r=0}^{\infty} e^{-\pi r^2} \, r\,dr \right) d\theta$$

$$u = \pi r^2$$

$$du = 2\pi r\,dr$$

Aha!

$$r\,dr = -\frac{1}{2\pi}\,du$$

when $r=0$, $u=0$

$r \to \infty$, $u \to \infty$.

$$\int_{r=0}^{\infty} e^{-\pi r^2} r\,dr = \int_{0}^{\infty} e^{-u} \frac{1}{2\pi}\,du$$

$$= \frac{1}{2\pi}\left[-e^{-u} \right]_0^\infty = \frac{1}{2\pi}$$

$$\text{Integral} = \int_{\theta=0}^{2\pi} \frac{1}{2\pi}\, d\theta = 1 \checkmark$$

Approx to identity

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\varepsilon x)$$