Lecture 33 Fourier Transform on $\mathcal{S}$

$$\mathcal{M}(\mathbb{R}) = \{ f : \text{continuous on } \mathbb{R} \text{ and } |f(x)| \le \frac{M}{1+x^2} \}$$

Remark: Safe to ignore $\int_{|x| > \text{Big}}$ for fcns in $\mathcal{M}$.

EX: $$\underbrace{\int_{-\infty}^{\infty} x \, dx}_{\infty - \infty} \overset{?}{=} \underbrace{\lim_{N\to\infty} \underbrace{\int_{-N}^{N} x \, dx}_{0}}_{\text{Principal value of } \int_{-\infty}^{\infty} x\,dx} = 0$$

For $f \in \mathcal{M}$: $$\int_{-\infty}^{\infty} f \, dx = \int_{-\infty}^{0} f \, dx + \int_{0}^{\infty} f \, dx$$

$$= \lim_{A\to-\infty}\int_A^0 f\,dx + \lim_{B\to\infty}\int_0^B f\,dx$$

$$= \lim_{N\to\infty}\int_{-N}^{N} f\,dx$$

Properties of $\int_{-\infty}^{\infty}$ on $f \in \mathcal{M}(\mathbb{R})$

1) $\int_{-\infty}^{\infty} f\,dx$ is <u>linear</u>.

2) $$\int_{-\infty}^{\infty} f(x+h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

3) $$\int_{-\infty}^{\infty} f(Kx)\,dx = \frac{1}{K}\int_{-\infty}^{\infty} f(x)\,dx$$

4) $$\int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx \longrightarrow 0 \text{ as } h\to 0.$$

<u>Pf</u> of 2): $I = \int_A^B f(\underbrace{x+h}_{u})\,dx$ $\qquad du = dx$

when $x = A$: $u = A+h$

$x = B$: $u = B+h$

$$I = \int_{A+h}^{B+h} f(u)\,du \longrightarrow \int_{-\infty}^{\infty} f\,dx$$

as $A \to -\infty$ and $B \to \infty$.

4) Let $\varepsilon > 0$. $f \in \mathcal{M}$.

$\exists R$ such that

$$\int_{|x|>R-1} |f|\,dx < \frac{\varepsilon}{4}.$$ Assume $|h|<1$.

f unif cont. on $[-R-1, R+1]$.

So $\exists\, \delta > 0$ such that

$$\left| f(x+h) - f(x) \right| < \frac{\varepsilon}{4R} \quad \text{if } |h| < \delta < 1.$$

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)|\, dx$$

$$< \underbrace{\int_{|x|>R} |f(x+h)|\, dx}_{< \frac{\varepsilon}{4}} + \underbrace{\int_{|x|>R} |f(x)|\, dx}_{< \frac{\varepsilon}{4}}$$

$$+ \underbrace{\int_{-R}^{R} |f(x+h) - f(x)|\, dx}$$

$$< \frac{\varepsilon}{4R} \cdot 2R = \frac{\varepsilon}{2}$$

Gaussian $\varphi(x) = e^{-\pi x^2}$

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$$

Today: will show $\hat{\varphi} = \varphi$

$\varphi \in \mathcal{S} \leftarrow$ Shawrtz Space of fcns rapidly decreasing fcns at ∞.

$f \in \mathcal{S}$ means:

$$\lim_{\substack{x\to\infty \\ x\to-\infty}} |f^{(n)}(x)|\,|x|^N = 0$$

for each n and $N \in \mathbb{N}$.

or equivalently

$$\sup_{x\in\mathbb{R}} |f^{(n)}(x)\,x^N| < \infty$$

for every $n, N \in \mathbb{N}$.

Facts: $f \in \mathcal{S} \Rightarrow f' \in \mathcal{S}$

$f \in \mathcal{S} \Rightarrow xf \in \mathcal{S}$

So $f \in \mathcal{S} \Rightarrow$ (polys) $\cdot$ derivatives of f stay in $\mathcal{S}$.

Big Fact: $f \in \mathcal{S} \Rightarrow \hat{f} \in \mathcal{S}$.

Important Properties of F. T. on $\mathcal{S}$:

$$f(x) \longrightarrow \hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s}\, dx$$

$\in \mathcal{S}$ $\qquad \in \mathcal{S}$

1) $f(x+h) \longrightarrow e^{2\pi i s h}\, \hat{f}(s)$

2) $f(x)e^{-2\pi i x h} \longrightarrow \hat{f}(s+h)$

3) $f(\delta x) \longrightarrow \frac{1}{\delta}\, \hat{f}\left(\frac{s}{\delta}\right)$

4) $f'(x) \longrightarrow 2\pi i s\, \hat{f}(s)$

5) $-2\pi i x f(x) \longrightarrow \hat{f}'(s)$

Use these to show that $\varphi = \hat{\varphi}$:

$$F(s) = \hat{\varphi}(s) = \int_{-\infty}^{\infty} \underbrace{e^{-\pi x^2}}_{\varphi(x)}\, e^{-2\pi i x s}\, dx$$

$$F'(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} \left[-2\pi i x \, e^{-2\pi i x s} \right] dx$$

$$= i \int_{-\infty}^{\infty} \underbrace{\left[-2\pi x \, e^{-\pi x^2} \right]}_{\varphi'(x)} e^{-2\pi i x s} \, dx$$

Aha!

$$= i \, \widehat{\varphi'} = i \left[2\pi i s \underbrace{\hat{\varphi}(s)}_{F(s)} \right]$$

$$\boxed{F'(s) = -2\pi s F(s)}$$

ODE

$$\frac{F'(s)}{F(s)} = -2\pi s$$

$$\frac{d}{ds} \ln F(s) = \frac{d}{ds}\left(-\pi s^2\right)$$

$$\frac{d}{ds}\Big[\underbrace{\ln F(s) + \pi s^2}_{\text{const. } K}\Big] \equiv 0$$

$$\ln F = K - \pi s^2$$

$$e \qquad e$$

$$F(s) = e^K e^{-\pi s^2}$$

Hmmm. $F(0) = e^K$

$$\parallel$$

$$\hat{\varphi}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} \underbrace{e^{-2\pi i x \cdot 0}}_{1} dx$$

$$= 1. \checkmark$$

So $F(s) = e^{-\pi s^2}$. $\hat{\varphi} = \varphi$ $\checkmark$

<u>Way to think about F.T. on $\mathcal{S}$</u>

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$$

$$f = \lim_{\varepsilon \to 0} \varphi_\varepsilon * f$$

$$\varphi_\varepsilon * f = \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(x-t)\, f(t)\, dt}_{\approx \text{ Riemann sum}}$$

$$\approx \underbrace{\sum c_i \varphi_\varepsilon(x - t_i)}_{\text{Finite sum}}$$

Philosophy: Know Fourier Inversion formula for $\varphi_\varepsilon(x-t_i)$.

Take limits in $\mathcal{S}$ to get F. I. F. for $f \in \mathcal{S}$.

Pf of $f(x+h) \to e^{2\pi i s h} \hat{f}(s)$

$$\int_{-\infty}^{\infty} f(x+h)\, e^{-2\pi i x s}\, dx$$

replace x by $x-h$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i (x-h) s}\, dx$$

$$= e^{2\pi i h s} \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s}\, dx}_{\hat{f}(s)} \quad \checkmark$$

Pf that $\widehat{f'} = 2\pi i s \hat{f}$:

$$\int_{-\infty}^{\infty} f'(x)\, e^{-2\pi i x s}\, dx =$$

$$\lim_{N \to \infty} \int_{-N}^{N} \underbrace{e^{-2\pi i x s}}_{u}\, \underbrace{f'(x)\, dx}_{dv}$$

$$u = e^{-2\pi i x s} \qquad du = -2\pi i s\, e^{-2\pi i x s}\, dx$$

$$v = f(x) \qquad dv = f'(x)\, dx$$

$$= \lim_{N \to \infty} \left[uv \Big|_{-N}^{N} - \int_{-N}^{N} v\, du \right]$$

$$= \lim_{N \to \infty} \left[\underbrace{e^{-2\pi i x s} f(x) \Big|_{-N}^{N}}_{\to 0} + \int_{-N}^{N} f(x) \left[2\pi i s\, e^{-2\pi i x s}\, dx \right] \right]$$

$$= 2\pi i s \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s}\, dx}_{\hat{f}(s)} \quad \checkmark$$