Lecture 34 The Fourier Inversion Formula

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{2\pi i s x}\, ds$$

where $$\hat{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx$$

for $f \in \mathcal{S}$: $\sup_{\mathbb{R}} |f^{(n)}(x)\, x^m| < \infty$

for each m, n.

(Same as $x^m f^{(n)}(x) \longrightarrow 0$ as $x \to \infty$, $x \to -\infty$

for each n, m.)

<u>Note</u>: $x^m f(x) \longrightarrow 0$

$$|f(x)| < \frac{c}{|x|^m} \quad \text{for } |x| > R$$

So $\int_{-\infty}^{\infty}$ exists and tail ends are <u>very</u> small.

Fact: $\tilde{\mathcal{F}}[f] = \hat{f}$

$\tilde{\mathcal{F}} : \mathcal{S} \to \mathcal{S}$

Why: $\left|\hat{f}(s)\right| = \left|\int_{-\infty}^{\infty} f(x) \underbrace{e^{-2\pi i s x}}_{\substack{\text{modulus} \\ =1}} dx\right|$

$$\leq \int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

So $\hat{f}$ is bdd for $f \in \mathcal{S}$

Key: $\tilde{\mathcal{F}}[x f(x)] = c\, \hat{f}'(s)$

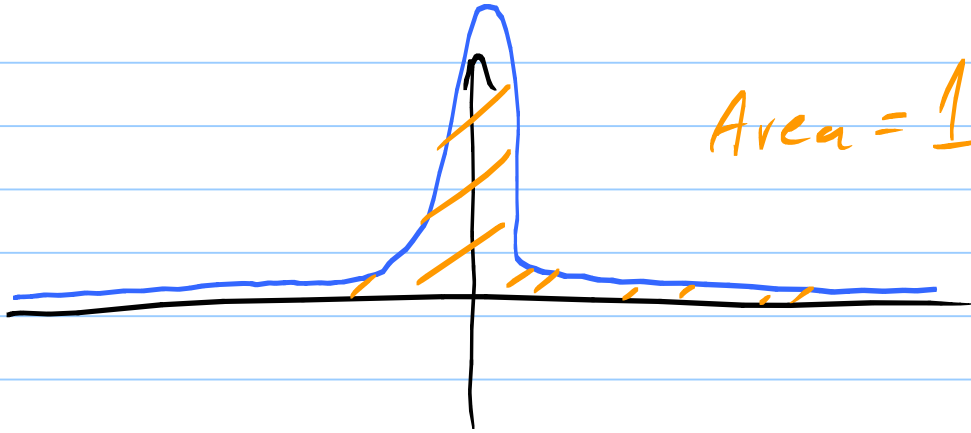
$$\tilde{\mathcal{F}}[f'(x)] = c\, s\, \hat{f}(s)$$

Repeat.

Gaussian: $\varphi(x) = e^{-\pi x^2}$

$$\int_{-\infty}^{\infty} \varphi\, dx = 1$$

"Good" kernel: $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$



<u>Lemma</u>: $f \in \mathcal{S}$. Then

$$\varphi_\varepsilon * f \longrightarrow f \quad \text{unif on } \mathbb{R}$$

as $\varepsilon \to 0$.

<u>Pf</u>: $(\varphi_\varepsilon * f)(x) - f(x) =$

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(x-t)\, f(t)\, dt \;-\; f(x) \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, dt}_{=1}$$

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(t)\left[f(x-t) - f(x)\right] dt$$

Let $\rho > 0$. $|f| < \frac{\rho}{2}$ if $|x| > R$.

f unif cont on $[-R, R]$.

Choose R big enough that

$$\int_{|t|>R} \varphi_\varepsilon(t)\,dt < \frac{\varepsilon}{2}, \quad \text{etc.}$$

$$\int_{|t|>R} \varphi_\varepsilon(t) \underbrace{\left[f(x-t) - f(x) \right]}_{|\quad| < 2M} dt \qquad |f| < M \text{ on } \mathbb{R}$$

$$+ \int_{-R}^{R} \varphi_\varepsilon(t) \left[f(x-t) - f(x) \right] dt$$

$$= \left(\int_{-R}^{-\tilde{\varepsilon}} + \int_{\tilde{\varepsilon}}^{R} \right) \underset{\text{small}}{\varphi_\varepsilon(t)} \underbrace{\left[f(x-t) - f(x) \right]}_{\text{bndd}} dt$$

$$\underbrace{+ \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \varphi_\varepsilon(t) \underbrace{\left[f(x-t) - f(x) \right]}_{\text{small}} dt}_{\le \text{ small } \int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \varphi_\varepsilon\,dt \,<\, \int_{-\infty}^{\infty} \varphi_\varepsilon\,dt \,=\, 1}$$

Lemma: $f, g \in \mathcal{S}$:

$$\int_{-\infty}^{\infty} f \hat{g}\, dx = \int_{-\infty}^{\infty} \hat{f} g\, dx$$

Pf: Fubinni on $f(x)g(y)e^{-2\pi i x y}$

$$\iint_{\mathbb{R}^2} f(x)g(y)\, e^{-2\pi i x y}\, dA$$

$$= \int_{-\infty}^{\infty} f(x) \Big(\underbrace{\int_{-\infty}^{\infty} g(y)\, e^{-2\pi i x y}\, dy}_{\hat{g}(x)} \Big) dx$$

$$= \int_{-\infty}^{\infty} g(y) \Big(\underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x y}\, dx}_{\hat{f}(y)} \Big) dy$$

Change dummy var. y to x. ✓

Fourier Inversion Theorem: $f \in \mathcal{S}$

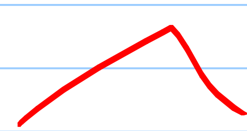
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{2\pi i x s}\, ds$$

Pf: Step 1: True at $x = 0$.

Want $$f(0) = \int_{-\infty}^{\infty} \hat{f}(s)\, ds$$

$$\int_{-\infty}^{\infty} f(x)\, \varphi_\varepsilon(x)\, dx = (\varphi_\varepsilon * f)(0)$$

$f(0)$



$\varphi(\varepsilon x)$

Recall $\hat{\varphi} = \varphi$.

Rule: $f(\delta x) \longrightarrow \underbrace{\frac{1}{\delta}\, \hat{f}\left(\frac{s}{\delta}\right)}$

$$\varphi(\varepsilon x) \longrightarrow \frac{1}{\varepsilon}\varphi\left(\frac{s}{\varepsilon}\right)$$

$$\uparrow$$

$$\hat{\varphi} = \varphi$$

$$f(0) \leftarrow \int_{-\infty}^{\infty} f(x) \underbrace{\varphi_\varepsilon(x)}_{\widehat{\varphi(\varepsilon x)}} \, dx \underset{\text{Lemma}}{=} \int_{-\infty}^{\infty} \hat{f}(s)\varphi(\varepsilon s)\, ds$$

$$= \int_{-\infty}^{\infty} \hat{f}(s) \underbrace{e^{-\pi \varepsilon^2 s}}_{\xrightarrow{\text{unif}} 1} \, ds$$

$$= \int_{-\infty}^{\infty} \hat{f}(s)\, ds \quad \checkmark$$

Rule: $f(x+h) \longrightarrow e^{2\pi i s h}\hat{f}(s)$

Let $F(y) = f(y+x)$.

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(s)\, ds$$

$$= \int_{-\infty}^{\infty} e^{2\pi i s x} \hat{f}(s)\, ds$$

Done!

Cor: $\mathcal{F} : \mathcal{S} \xrightarrow[\text{onto}]{1\text{-}1} \mathcal{S}$ linear

Cosine Transform:

$$\underbrace{f(x)}_{\text{real valued}} = \int_{-\infty}^{\infty} \hat{f}(s) \left[\cos 2\pi s x + i \sin 2\pi s x \right] ds$$

$$\hat{f}(s) = \int_{-\infty}^{\infty} \underset{\text{even}}{f(x)} \left[\underset{\text{even}}{\cos 2\pi s x} - i \underset{\text{odd}}{\sin 2\pi s x} \right] dx$$

Given f on $\mathbb{R}^+$, extend it to $\mathbb{R}$ as an even fcn.

Aha! $\hat{f}(s) = \int_{-\infty}^{\infty} f(x) \cos 2\pi s x\, dx$

real valued $\longrightarrow$ $= 2 \int_0^{\infty} f(x) \cos 2\pi s x\, dx$

a Cosine transform!

Now

$$f(x) = \int_{-\infty}^{\infty} \left[\text{real valued}\right] \left(\cos 2\pi s x + i \sin 2\pi s x\right) ds$$

real $= A + Bi$

B must $= 0$.

$$f(x) = \int_{-\infty}^{\infty} \underbrace{\left(\begin{matrix}\text{Cosine Transf} \\ f\end{matrix}\right)}_{\text{even}} \cos 2\pi s x \, ds$$

$$= 2 \int_{0}^{\infty}$$

Aha! $\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c = id$

Fourier Sine Transf : Repeat with odd extension.

Last thing. Take $f(x)$ real valued

F. Inversion in terms of Cos + i Sin

Write it out, multiply it out. Take real part. Get classic Fourier Integral.