Lecture 35 Plancherel Identity

Remark: $f \in \mathcal{S}$, $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$

where $\varphi(x) = e^{-\pi x^2}$.

$\varphi_\varepsilon * f \to f$ unif on $\mathbb{R}$.

To prove the Fourier Inversion Formula, only needed:

$$\int_{-\infty}^{\infty} f(x)\,\varphi_\varepsilon(x)\,dx \longrightarrow f(0)$$

Easier:

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(x)\left[f(x) - f(0)\right]dx$$

$$\int_{|x|>R} \varphi_\varepsilon(x)\underbrace{\left[f(x) - f(0)\right]}_{\text{bndd by } 2M}dx \;+$$

$\int_{|x|>R} \varphi_\varepsilon\,dx$ small

$$\left(\int_{-R}^{-\tilde{\varepsilon}} + \int_{\tilde{\varepsilon}}^{R}\right) \underbrace{\varphi_\varepsilon(x)}_{\substack{\to 0 \\ \text{unif} \\ \text{as } \varepsilon \to 0}} \underbrace{[f(x) - f(0)]}_{\substack{\text{bndd} \\ \text{by } 2M}} dx \; +$$

$$\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \underbrace{\varphi_\varepsilon(x)}_{\uparrow} \underbrace{[f(x) - f(0)]}_{\text{small}} dx$$

$$\int_{-\tilde{\varepsilon}}^{\tilde{\varepsilon}} \varphi_\varepsilon(x)\,dx < \int_{-\infty}^{\infty} \varphi_\varepsilon(x)\,dx = 1$$

Plancherel Identity: $f \in \mathcal{S}$

$$\|f\| = \|\hat{f}\|$$

where $\|f\| = \sqrt{\int_{-\infty}^{\infty} |f|^2\,dx}$

[~ Parseval's Identity]

<u>Key</u>: $$\int_{-\infty}^{\infty} f\,\hat{g}\,dx = \int_{-\infty}^{\infty} \hat{f}\,g\,dx$$

for $f, g \in \mathcal{S}$.

Hmmm. What if $\hat{g} = \bar{f}$?

Then $g = \check{\bar{f}}$ ← Fourier Inversion Formula

$$= \int_{-\infty}^{\infty} \overline{f(x)}\, \underbrace{e^{2\pi i s x}}_{= \overline{e^{-2\pi i s x}}}\, dx$$

$$\boxed{\check{\bar{f}} = \overline{\hat{f}}}$$

$$= \overline{\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx}$$

$$= \overline{\hat{f}(s)}$$

$$\int_{\mathbb{R}} f \cdot \underbrace{\bar{f}}_{\hat{g}}\, dx = \int \hat{f}\, \underbrace{\overline{\hat{f}}}_{g}\, dx \quad \checkmark$$

Exciting consequence :

$$C_0^\infty(\mathbb{R}) \subset \mathcal{S} \subset L^2(\mathbb{R})$$

dense subspace of $L^2(\mathbb{R})$

Pf: $f \in L^2(\mathbb{R})$

Step 1 : $f_N = \chi_{[-N,N]} \cdot f$

$$f_N \to f \text{ in } L^2$$

Step 2 : ψ bump fcn in $C_0^\infty(-1,1)$

$$\underbrace{\psi_\varepsilon * f_N}_{\in C_0^\infty(\mathbb{R})} \to f_N \text{ in } L^2$$

Given $f \in L^2(\mathbb{R})$, get seq $f_n \in C_0^\infty(\mathbb{R})$

with $f_n \to f$ in L^2, meaning that

$$\|f_n - f\| \to 0.$$

Then f_n's form a Cauch seq: Given $\varepsilon > 0$,

$\exists N$ s.t. $\|f_n - f_m\| < \varepsilon$. if $n, m > N$.

Plancherel: $\|\hat{f}_n - \hat{f}_m\| < \varepsilon$ if $n, m > N$

So $\hat{f}_n$ are Cauchy!

$L^2(\mathbb{R})$ is a complete metric space with respect to $\| \ \|$. So $\hat{f}_n \to F \in L^2$.

$$f_n \to f, \qquad \hat{f}_n \to F$$

def^n of $\hat{f}$

Fact: $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Hilbert space isometry.

$$\|f\| = \|\hat{f}\| \implies$$

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

<u>Distributions</u>: Continuous linear functionals on $C_0^\infty(\mathbb{R})$.

<u>Tempered distributions</u>: Cont. lin. fcn's on $\mathcal{S}$.

$f_n \in \mathcal{S}$, $f \in \mathcal{S}$.

$f_n \to 0$ in $\mathcal{S}$ means

$$\sup_{\mathbb{R}} |f_n^{(k)}(x)|\,|x^m| \to 0$$

as $n \to \infty$ for each k, m.

$f_n \to f$ means $f_n - f \longrightarrow 0$.

$\lambda : \mathcal{S} \to \mathbb{R}$ is a <u>T. distribution</u>

1) linear

2) If $f_n \to f$ in $\mathcal{S}$, then

$$\lambda(f_n) \to \lambda(f).$$ ← continuous

Generalizations of fcns:

How to think of a continuous fcn f as a T. distribution: λ_f

$$\lambda_f(\varphi) = \int_{\mathbb{R}} f \varphi \, dx$$

Note: f, g cont and $\lambda_f = \lambda_g$, then $f \equiv g$. [Because of C_0^∞ bump fcns]

Delta fcn: δ

$$\lambda_\delta(\varphi) = \varphi(0)$$

Gaussians: $\varphi_\varepsilon \longrightarrow \delta$ "in the sense of distributions"

$$\int \varphi_\varepsilon \psi \, dx \underset{\varepsilon \to 0}{\longrightarrow} \psi(0)$$

$$\lambda_\delta(\psi) = \int \delta \psi \, dx = \psi(0)$$

Derivatives in the sense of distributions

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \varphi \, dx = - \int_{-\infty}^{\infty} f \frac{\partial \varphi}{\partial x} \, dx$$

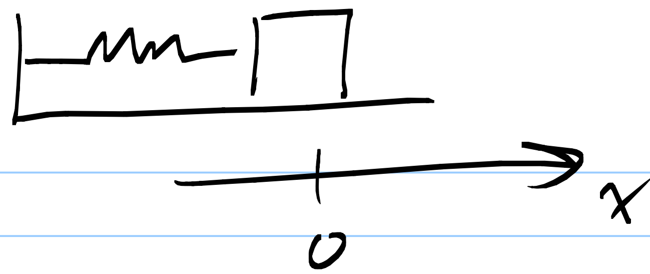
think f nice and diff'ble.

$\frac{\partial f}{\partial x} = g$ in sense of dist if

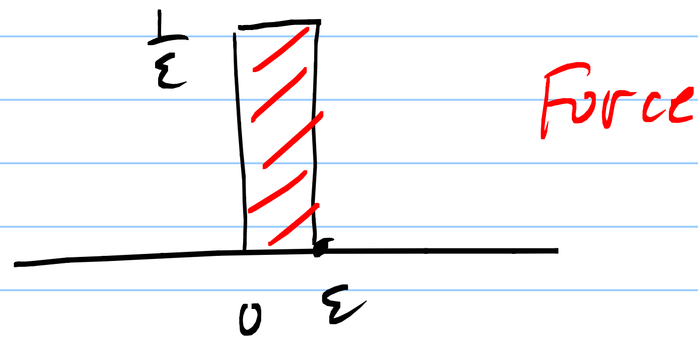
$$\int g \varphi \, dx = - \int f \frac{\partial \varphi}{\partial x} \, dx$$

for all $\varphi \in \mathcal{S}$.

Delta fcn:

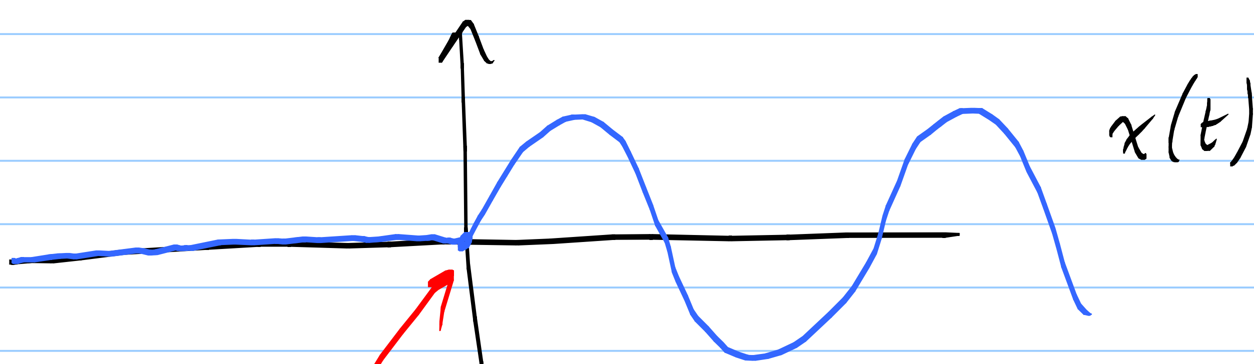


$m\ddot{x} + kx =$



As $\varepsilon \to 0$, Force fcn $\longrightarrow$ Perfect sledge hammer impulse = delta fcn

Even though right hand side is not a fcn, the x_ε sol^{n}s $\longrightarrow x$, a true fcn



hits

Get a unit increase in momentum.