Lecture 36 Poisson Summation Formula

Fourier Integral convolution : $f, g \in \mathcal{S}$

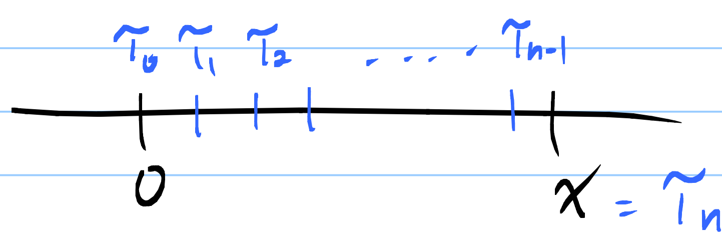
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

Facts : $f * g = g * f$

$$\widehat{f*g} = \hat{f}(s)\, \hat{g}(s)$$

Note : Laplace Transform convolution :

$$(f *_L g)(x) = \int_0^x f(\tau)\, g(x-\tau)\, d\tau$$



$$\mathcal{L}[f *_L g] = F(s)\, G(s)$$

$\mathcal{L}[f] = F(s)$

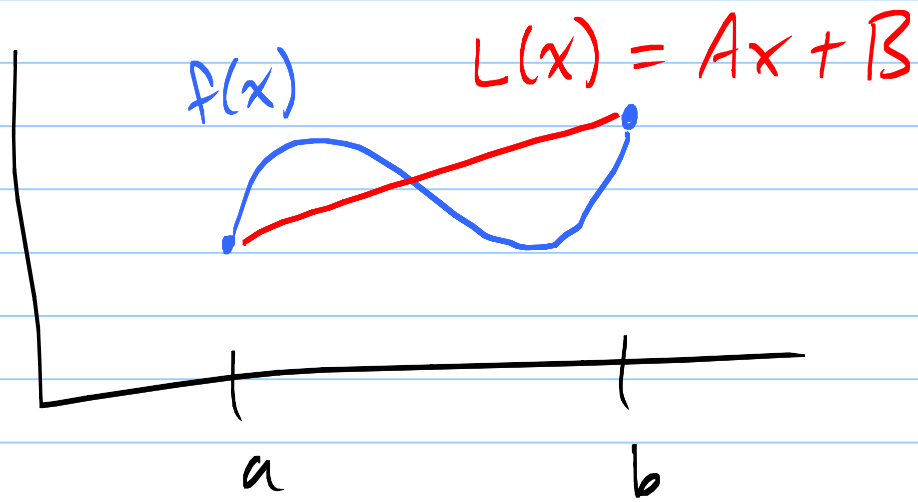
$\mathcal{L}[g] = G(s)$

Remark : Basic Facts about F.T.

extend routinely from fcns in $\mathcal{S}$ to fcns of moderate decrease whose F.T. is also of mod. dec.

Stein: Weierstraß Approx. Thm.

f continuous on $[a,b]$.



Note: If I can approx $f - L \approx$ poly.

Then $f \approx L +$ poly $=$ poly.

Hence: Can assume $f(a) = 0 = f(b)$.

Gaussian $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} e^{-\pi (x/\varepsilon)^2}$

$\varphi_\varepsilon * f \longrightarrow f$ unif on $\mathbb{R}$.

Restrict attention to $[-M, M]$ that contains $[a,b]$.

Approximate φ_ε by beginning poly of the power series for φ_ε.

$$P(x) \approx \varphi_\varepsilon.$$

$$\varphi_\varepsilon * f \approx (P * f)(x)$$

$$= \int_{-M}^{M} \underbrace{P(x-t)}_{\sum_0^N c_n (x-t)^n} f(t)\, dt$$

$$= \text{Sum } x^m \int (\text{Poly in } t)\, f(t)\, dt$$

Poisson Summation Formula $f \in \mathcal{S}$

$$\underbrace{\sum_{n=-\infty}^{\infty} f(x+n)}_{C^\infty \text{ fcn}} = \sum_{n=-\infty}^{\infty} \underbrace{\hat{f}(n)}_{} e^{2\pi i n x}$$

F.T. of f at n

periodic with period 1

Case $x=0$: $$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \quad !$$

Famous Formula:

$$\frac{\pi^2}{\sin^2 \pi \alpha} = \sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} \qquad \alpha \notin \mathbb{Z}$$

Pf: of Poisson Summation.

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) \leftarrow \text{Period} = 1, \; C^\infty$$

$$\hat{F}(m) = \int_0^1 F(x) e^{-2\pi i m x} \, dx$$

Fourier Series coeff!

$$= \int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n) e^{-2\pi i m x} \right) dx$$

$$= \sum_{n=-\infty}^{\infty} \left(\int_0^1 f(\underbrace{x+n}_{\tau}) e^{-2\pi i m x} \, dx \right)$$

(because of unif conv

$dx = d\tau$ $\quad$ $x = \tau - n$

When $x=0$: $\tilde{\tau}=0+n=0$

$x=1$: $\tilde{\tau}=1+n$

$$= \sum_{n=-\infty}^{\infty} \left(\int_{n}^{n+1} f(\tilde{\tau}) \underbrace{e^{-2\pi i m(\tilde{\tau}-n)}}_{= e^{-2\pi i m \tilde{\tau}} \underbrace{e^{2\pi i m n}}_{=1}} d\tilde{\tau} \right)$$

$$= \int_{-\infty}^{\infty} f(\tilde{\tau}) e^{-2\pi i m \tilde{\tau}} d\tilde{\tau} = \hat{f}(m)$$

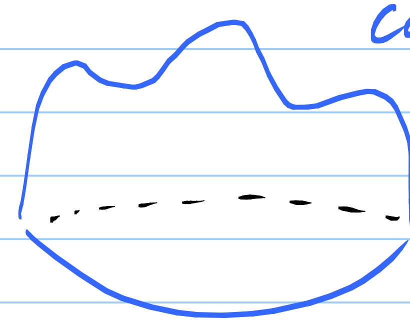
$s=m$

Fourier coeff for $F(x)$ is $\hat{f}(m)$

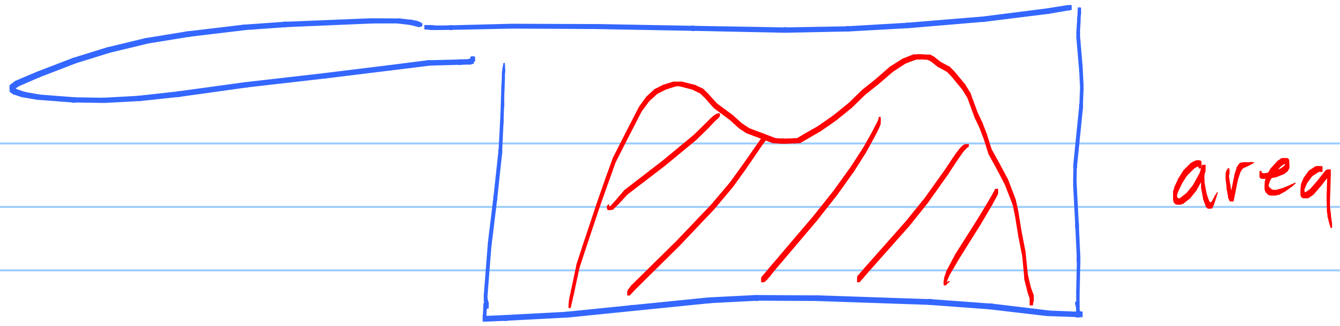
and the Fourier series converges nicely to $F(x)$. ✓

Radon Transform:

Birthday cake



Cut with a cleaver



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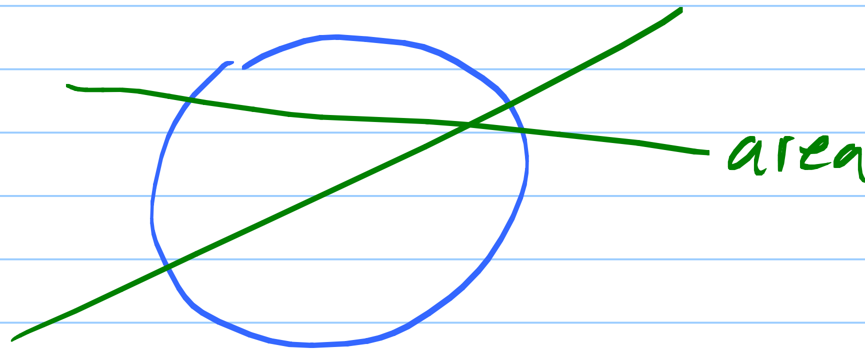
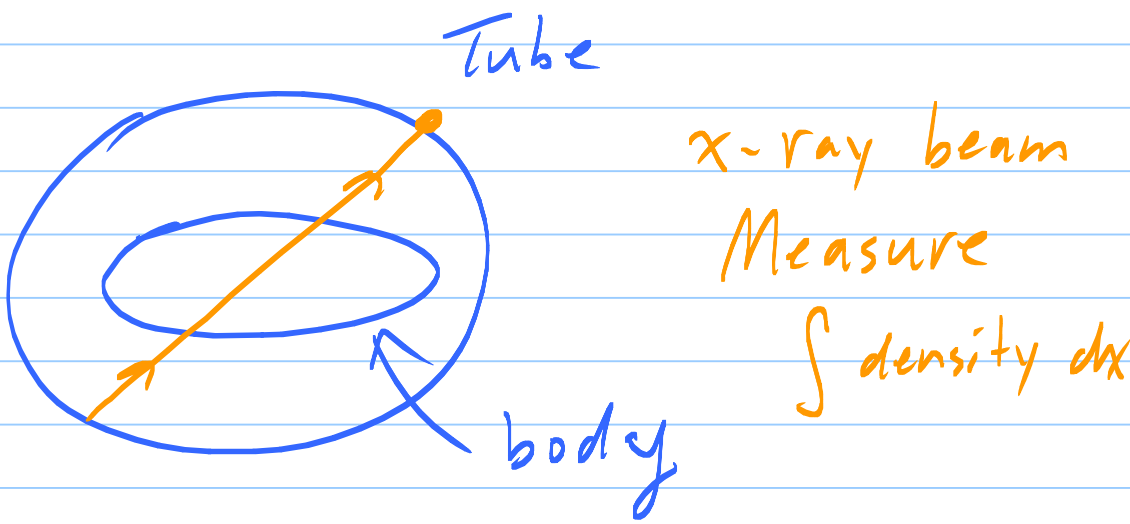


Radon Transform:

$\underbrace{\text{fcn}}_{\text{cake shape}} \longmapsto$ data: lines, areas.



Problem: Can you recover the density

fcn from the $\int \delta\, dx$ along lines?

Exciting fact: Fourier inversion formula can be used in a straigtforward way to get the Radon Inversion Formula, the basis of X-ray tomography.