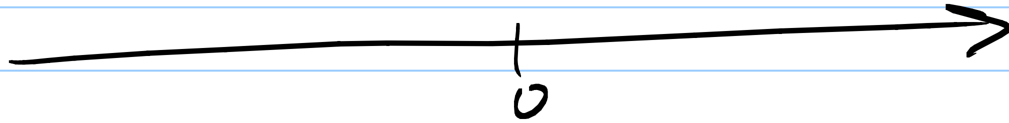
Lecture 37: Heat problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{hot wire}$$



$$u(x,0) = f(x) \in \mathcal{S}$$

$$\hat{u}(s,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-2\pi i x s}\, dx$$

$$\frac{\partial \hat{u}}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\, e^{-2\pi i x s}\, dx$$

$$= \widehat{\frac{\partial u}{\partial t}}$$

$$\mathcal{F}_x\left[\frac{\partial^2 u}{\partial x^2}\right] = \mathcal{F}_x\left[\frac{\partial u}{\partial t}\right] = \frac{\partial \hat{u}}{\partial t}$$

$$\|$$

$$(2\pi i)s \cdot (2\pi i)s\ \hat{u}(s,t)$$

$$-4\pi^2 s^2 \, \hat{u}(s,t) = \frac{\partial}{\partial t} \hat{u}(s,t)$$

$$\hat{u}(s,t) = C(s)\, e^{-4\pi^2 s^2}$$

Plug in $t=0$:

$$\hat{u}(s,0) = C(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx$$

$$= \hat{f}(s).$$

So $C(s) = \hat{f}(s)$.

$$\hat{u}(s,t) = \hat{f}(s)\, e^{-4\pi^2 s^2 t}$$

Fourier Invert:

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(s)\, \underbrace{e^{-4\pi^2 s^2 t}\; e^{2\pi i s x}}_{\text{Satisfies heat eqn in } x,\, t}\, ds$$

$\hat{f} \in \mathcal{S}$

<u>Today</u>: Can diff'tiate under $\int$ because $\hat{f} \in \mathcal{S}$.

See $u \in C^\infty(\mathbb{R} \times \mathbb{R}^+)$ and u satisfies the heat eqn.

Hmmm: $$\hat{u}(s,t) = \hat{f}(s) \cdot \underbrace{e^{-4\pi^2 s^2 t}}_{\widehat{\varphi_\varepsilon(x)}}$$

where $\varepsilon = \sqrt{4\pi t}$

So $$\hat{u}(s,t) = \widehat{(f * \varphi_\varepsilon)}$$ and

$$u(x,t) = f * \varphi_\varepsilon$$

heat kernel

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon} e^{-\pi (x/\varepsilon)^2} \quad \text{where } \varepsilon = \sqrt{4\pi t}$$

$$\boxed{\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}}$$

Aha! We know $\varphi_\varepsilon * f = f * \varphi_\varepsilon$

$\longrightarrow f$ uniformly on $\mathbb{R}$

as $\varepsilon \to 0$. But $\varepsilon = \sqrt{4\pi t} \to 0$

as $t \to 0$.

$u(x,t) = f * \mathcal{H}_t \longrightarrow f$ unif as $t \to 0$.

$$u(x,t) = \int_{-\infty}^{\infty} f(x-y) \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t} \, dy$$

Not only does $u(x,t) \to f(x)$ unif in x as $t \to 0$, but

$$\int_{-\infty}^{\infty} |u(x,t) - f(x)|^2 \, dx \to 0 \quad \text{as } t \to 0$$

too.

|| Parseval's

$$\int_{-\infty}^{\infty} \underbrace{|\hat{u}(s,t) - \hat{f}(s)|^2}_{|\hat{f}(s)|^2 \underbrace{\left| e^{-4\pi^2 s^2 t} - 1 \right|^2}} ds$$

$$= \int_{|x|>R} + \int_{-R}^{R}$$

small because $\hat{f} \in \mathcal{S}$

$\to 0$ unif on $[-R, R]$ as $t \to 0$.

Solving 2nd order linear ODE via power series:

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = 0$$

where A, B, C = convergent power series.

Danger: Nasty spots where $A(x) = 0$.

__EX__: Airy's Eqn:

$$y'' - x\,y = 0$$

Try $\displaystyle y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + \cdots$$

Want $y'' - x\,y = 0$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{\underbrace{n-2}_{m}} - \sum_{n=0}^{\infty} a_n x^{n+1} \overset{\text{want}}{=} 0$$

Think change of vars in an $\int$.

$m = n-2$ When $n=2$, $m=0$

$n = m+2$ As $n\to\infty$, $m\to\infty$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} a_{m-1}x^m \overset{\text{want}}{=} 0$$

$$\underbrace{2\cdot 1\cdot a_2}_{\substack{\text{straggler}\\ \text{const term}}} + \sum_{n=1}^{\infty}\Big[\underbrace{(n+2)(n+1)a_{n+2} - a_{n-1}}_{\text{Taylor's } = \frac{f^{(n)}(0)}{n!}}\Big]x^n \overset{\text{want}}{=} 0$$

$f \equiv 0$

Recursion formula :

$$\boxed{2a_2 = 0}$$

$$\boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}}$$

$n = 1, 2, \ldots\ldots$

<u>n=1</u>: $a_3 = \frac{a_0}{3\cdot 2}$

$a_0 = y(0)$, $a_1 = y'(0)$ } I.C.!

<u>n=2</u>: $a_4 = \frac{a_1}{4\cdot 3}$

<u>n=3</u>: $a_5 = \frac{a_2}{5\cdot 4} = 0$ because $a_2 = 0$.

<u>n=4</u>: $a_6 = \frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2}$

<u>n=5</u>: $a_7 = \frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}$

<u>n=6</u>: $a_8 = 0$

$$y(x) = \sum_{1}^{\infty} a_n x^n = a_0\left(\qquad\right) + a_1\left(\qquad\right)$$

$$a_0\left[1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} + \frac{x^9}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} + \cdots\right]$$

$$+\ a_1\left[x+\frac{x^4}{4\cdot 3}+\frac{x^7}{7\cdot 6\cdot 4\cdot 3}+\cdots\right]$$

Claim: Radius of conv $= \infty$

These really do solve the ODE!

Ratio test: $\sum u_n$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{u_{\text{far out}}}{u_{\text{before it}}}\right|$$

$$= \left|\frac{a_{n+2}\,x^{n+2}}{a_{n-1}\,x^{n-1}}\right|$$

$$= \frac{1}{(n+2)(n+1)}\,|x|^3 \longrightarrow 0 < 1$$

as $n \to \infty$

from recursion

Ratio Test $\Rightarrow$ Airy fcns converge for all x and are C^∞ smooth and solve Airy's Eqn.