Lecture 39 Circular drum

R-prob: $\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} = \lambda$, $R(1) = 0$

Last time: $\lambda = 0$ case. Euler eqn.

$\lambda > 0$: No non-zero sol^{n}s.

Case $\lambda < 0$: Write $\lambda = -k^2$

$$r^2 R'' + rR' + k^2 r^2 R = 0$$
$$R(1) = 0$$

Change variables: $s = kr$

$$s^2 \frac{d^2R}{ds^2} + s\frac{dR}{ds} + s^2 R = 0$$

Bessel's Eqn of order 0.

Note: ODE is singular at $s = 0$.

It has a "Regular singular pt. at $s=0$."

Method of Frobenius:

$$x^2 y'' + x y' + x^2 y = 0$$

Euler eqn: Try $y = x^p$.

Frobenius: Try $x^2 \cdot \left[y = x^p \sum_{n=0}^{\infty} a_n x^n \right]$

$$x \cdot \left[y' = p x^{p-1} \sum_0^\infty a_n x^n + x^p \sum_1^\infty n a_n x^{n-1} \right]$$

$$x^2 \cdot \left[y'' = p(p-1) x^{p-2} \sum_0^\infty a_n x^n + 2p x^{p-1} \sum_1^\infty n a_n x^{n-1} + x^p \sum_2^\infty n(n-1) a_n x^{n-2} \right]$$

$+$

$$p(p-1) x^p \sum_0^\infty a_n x^n + 2p x^p \sum_1^\infty n a_n x^n + x^p \sum_2^\infty n(n-1) a_n x^n$$

$$+ p x^p \sum_0^\infty a_n x^n + x^p \sum_1^\infty n a_n x^n$$

$$+ \underbrace{x^p \sum_0^\infty a_n x^{n+2}}_{x^p \sum_2^\infty a_{n-2} x^n}$$

Collect $n=0$ terms:

$$\underbrace{\left[p(p-1)+p\right]}_{\text{must be zero}} x^p a_0 + \text{higher order terms}$$

Indicial Equation: $p(p-1)+p=0$

$$p^2=0$$

Usually get two p's! Only get one, namely $p=0$.

$$y = x^0 \sum_{n=0}^{\infty} a_n x^n \leftarrow \text{a regular series!}$$

$$x^2 \cdot \left[y = \sum_0^{\infty} a_n x^n \right] = \sum_0^{\infty} a_n x^{n+2} = \sum_2^{\infty} a_{n-2} x^n$$

$$x \cdot \left[y' = \sum_1^{\infty} n a_n x^{n-1} \right]$$

$$x^2 \cdot \left[y'' = \sum_2^{\infty} n(n-1) a_n x^{n-2} \right] \quad +$$

$$\sum_2^{\infty} n(n-1) a_n x^n + \sum_1^{\infty} n a_n x^n + \sum_2^{\infty} a_{n-2} x^n$$

$n=1$ straggler term

want

$$a_1 x + \sum_{n=2}^{\infty} \Big[\underbrace{\big(n(n-1)+n\big)a_n + a_{n-2}}_{\frac{f^{(n)}(0)}{n!} = 0} \Big] x^n = 0$$

$a_1 = 0$

$f \equiv 0$

Recursion Formula: $\boxed{a_n = \dfrac{-a_{n-2}}{n^2}}$ $n = 2, 3, 4, \ldots$

$y(0) = a_0$

$a_1 = 0$ from straggler term

<u>$n=2$</u>: $a_2 = \dfrac{-a_0}{2^2}$

<u>$n=3$</u>: $a_3 = \dfrac{-a_1}{3^2} = 0$ ← all odd terms $= 0$.

<u>$n=4$</u>: $a_4 = \dfrac{-a_2}{4^2} = \dfrac{a_0}{4^2 \cdot 2^2}$

<u>$n=5$</u>: $a_5 = \dfrac{-a_3}{5^2} = 0$

<u>$n=6$</u>: $a_6 = -\dfrac{a_4}{6^2} = -\dfrac{a_0}{6^2 \cdot 4^2 \cdot 2^2}$

$\underline{n=8}$:

$$a_8 = \frac{a_0}{8^2 \cdot 6^2 \cdot 4^2 \cdot 2^2}$$

$$= \frac{a_0}{(2\cdot 4)^2 \cdot (2\cdot 3)^2 \cdot (2\cdot 2)^2 \cdot (2\cdot 1)^2}$$

$$= \frac{a_0}{2^8\,(4!)^2}$$

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n}(n!)^2}$$

Bessel fcn of order zero

$$J_0(x) = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{4^2\cdot 2^2} - \frac{x^6}{6^2\cdot 4^2\cdot 2^2} + \cdots\right]$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Take $a_0 = 1$.

Got only one sol^n. J_0

Method of reduction order:

Got $y_1 = J_0(x)$,

Get $y_2 = u(x) J_0(x)$

Plug into ODE. Get a first order ODE in u to solve.

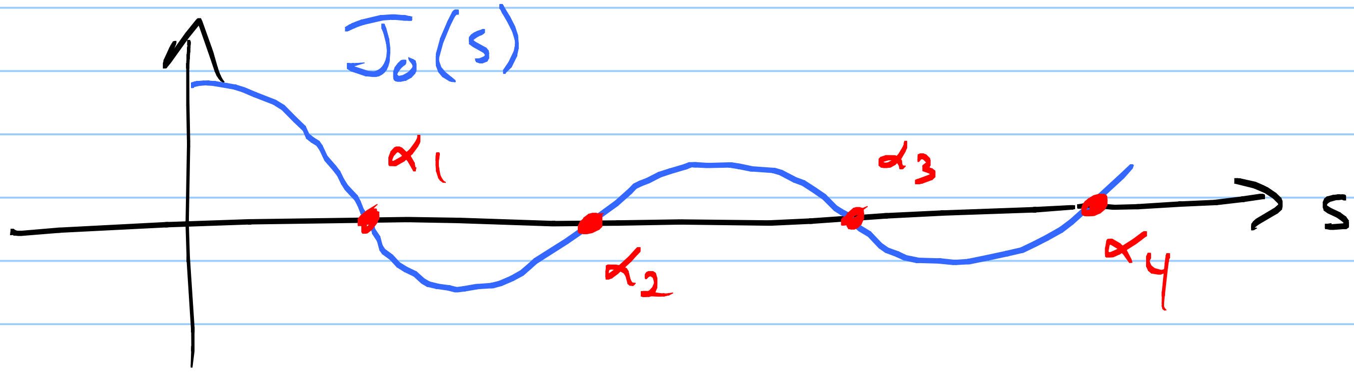
Do it. Find that y_2 is a drum ripper!

Get solution $y = J_0(x)$.

Back to R and r.

$$R(s) = J_0(s), \quad s = kr$$

$$R(r) = J_0(kr)$$



Need $R(1) = J_0(k \cdot 1) = 0$

Get non-zero solutions when

$K = \alpha_1, \alpha_2, \alpha_3, \ldots$

Back to T - prob:

$$T'' - \lambda T = 0 \qquad \left(\begin{matrix} \lambda = -K^2 \\ K = \alpha_n \end{matrix} \right)$$

$$T'' + K^2 T = 0$$

$$T_n(t) = A_n \cos \alpha_n t + B_n \sin \alpha_n t$$

$$u(r,t) = \sum_{n=1}^{\infty} \left(A_n J_0(\alpha_n r) \cos \alpha_n t + B_n J_0(\alpha_n r) \sin \alpha_n t \right)$$

$$u(r,0) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) \overset{\text{want}}{=} f(r)$$

$$\frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} \alpha_n B_n J_0(\alpha_n r) \overset{\text{want}}{=} g(r)$$

Theory of singular Sturm-Liouville probs:

$\varphi_n(r) = J_0(\alpha_n r)$ are orthogonal on $0 \le r \le 1$ with respect to weight fcn r.

$$\langle \varphi_n, \varphi_m \rangle = \int_0^1 \varphi_n(r)\varphi_m(r)\, r \, dr$$

weight fcn

Just like Fourier Series!