Lecture 4

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{Let } x = i\theta$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{\overset{-1}{i^2}\theta^2}{2!} + \frac{\overset{-i}{i^3}\theta^3}{3!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i \sin\theta$$

Complex exponential: $e^{x+iy} = e^x e^{iy}$

$$= e^x \cos y + i e^x \sin y$$

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \end{cases}$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\left(\frac{e^{inx}+e^{-inx}}{2}\right) + b_n\left(\frac{e^{inx}-e^{-inx}}{2i}\right)\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\left(\tfrac{1}{2}(a_n - ib_n)\right)e^{inx} + \left(\tfrac{1}{2}(a_n+ib_n)\right)e^{-inx}\right]$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

because
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

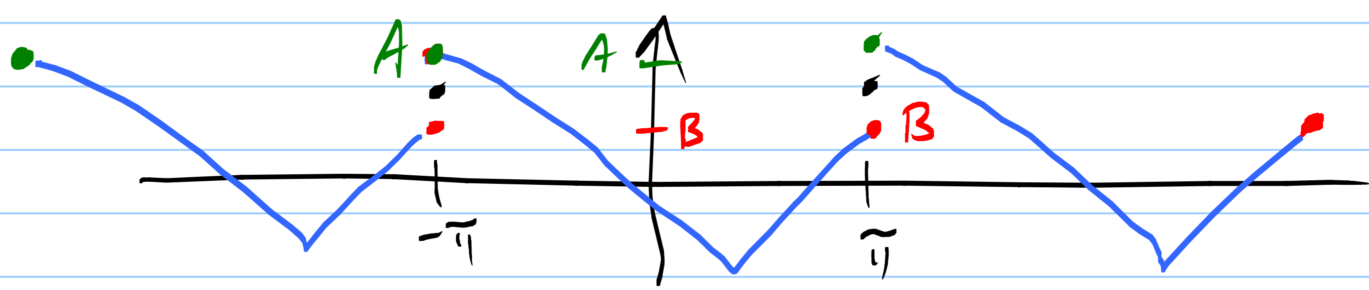
<u>Def^n</u>: Trigonometric poly of deg N:

$$P_N(x) = a_0 + \sum_{n=1}^{N}\left(a_n\cos nx + b_n\sin nx\right)$$

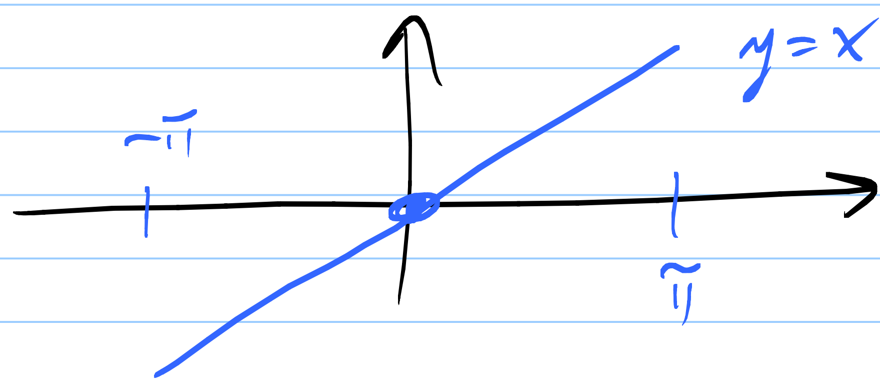
$$= \sum_{n=-N}^{N} c_n e^{inx}$$

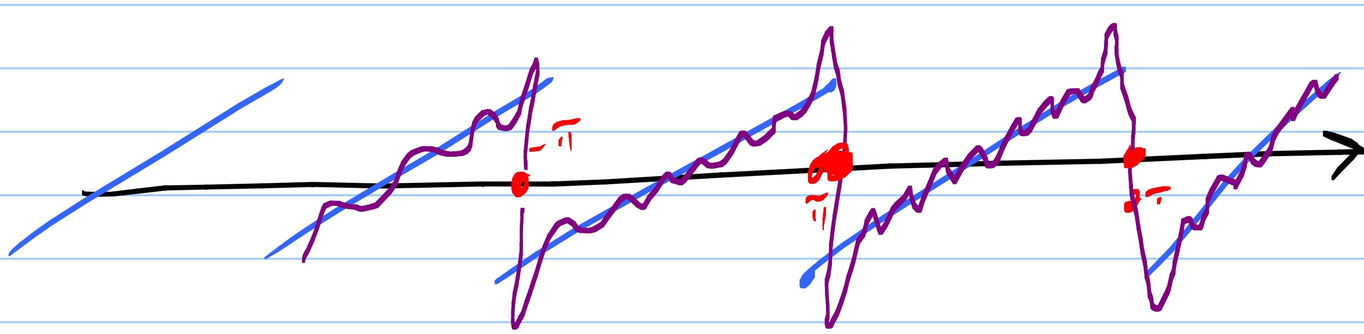
Why poly?: $\left(e^{inx}\right) = \left(e^{ix}\right)^n$

<u>Big Fact</u>: Fourier series converge to $f(x)$ swiftly on intervals where f is smooth, converges to $f(x)$ even at corners (but slowly), and to the <u>midpoint</u> of jumps, and to $\underline{\frac{A+B}{2}}$ at $x = \pm\pi$.



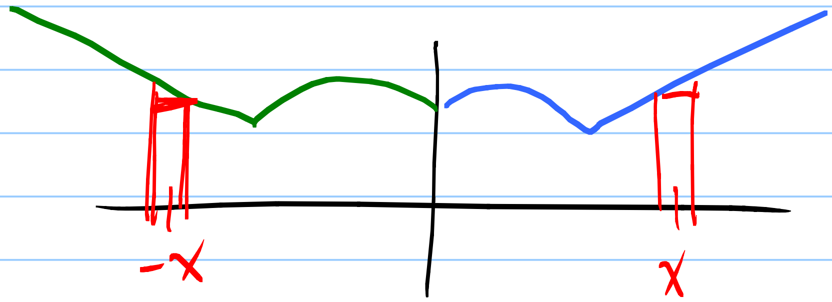
<u>EX</u>:





Even & odd fcns and F. Series:

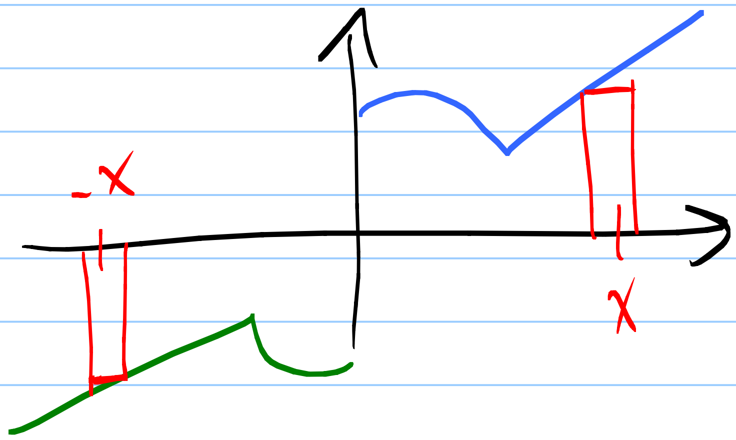
Even: $f(-x) = f(x)$



$$\int_{-L}^{0} f(x)\,dx = \int_{0}^{L} f(x)\,dx$$

So $$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

ODD: $f(-x) = -f(x)$



$$\int_{-L}^{0} f(x)\,dx = -\int_{0}^{L} f(x)\,dx$$

So $$\int_{-L}^{L} f(x)\,dx = 0$$

Facts: (EVEN)(EVEN) = EVEN

(EVEN)(ODD) = ODD

(ODD)(ODD) = EVEN

Why: EVEN $f(-x) = f(x)$

ODD $g(-x) = -g(x)$

$h(x) = f(x)g(x)$: $h(-x) = -h(x)$ ⟸ ODD

Consequently: 1, $\cos nx$ even

$\sin nx$ odd

EX: $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}}\ \underbrace{\sin nx}_{\text{odd}}}_{\text{odd}}\,dx = 0$$

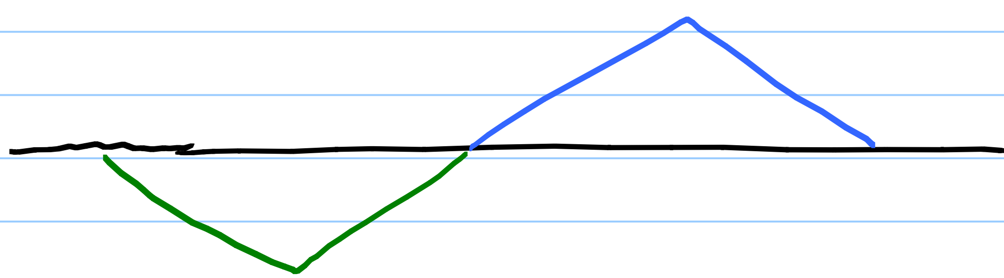
Fact: 1) ODD fcn has no Cosine or constant term in F. Series.

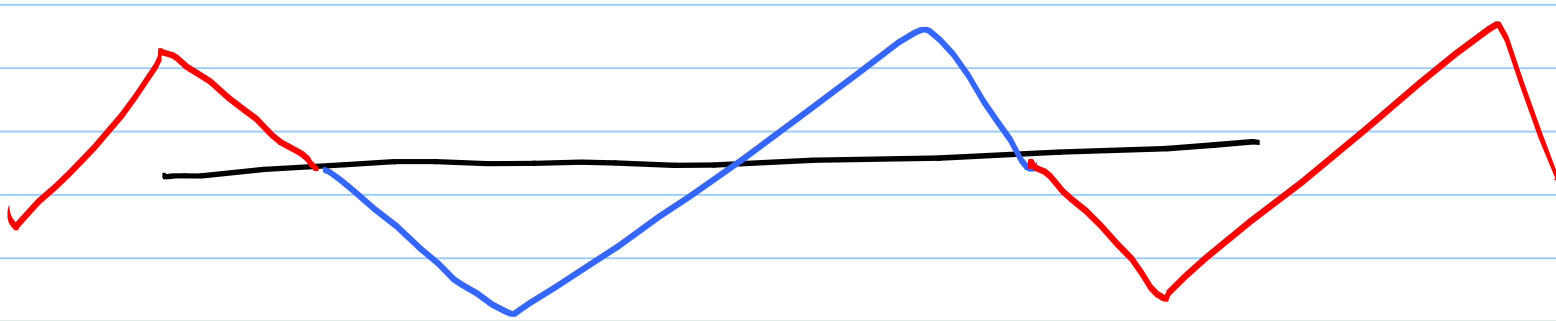
2) EVEN fcn has no Sine terms.

Fun fact: Any fcn can be expressed as (ODD) + (EVEN) fcn:

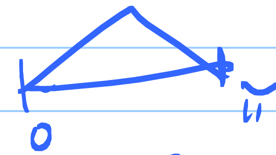
$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

EX: odd extension



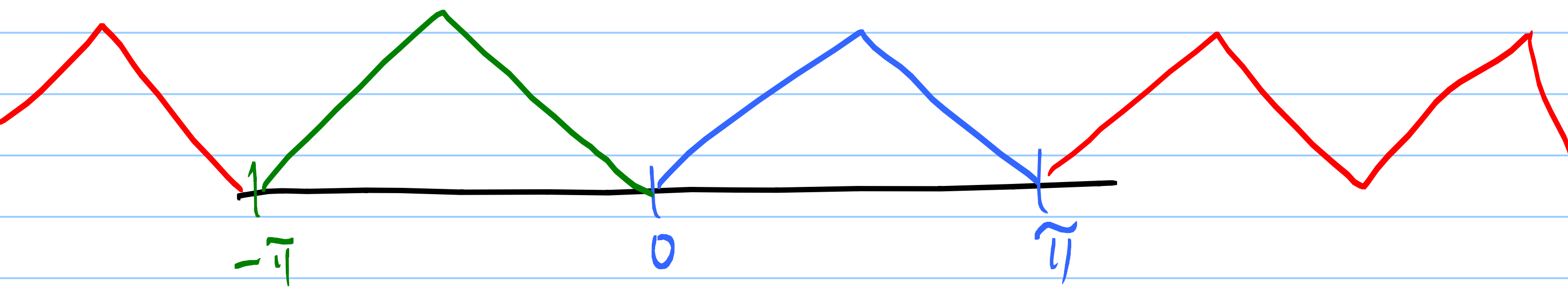


odd periodic extension

Aha! The full fourier series for the odd periodic extension of 

is the fourier sine series for the fcn on $[0,\pi]$.

Even periodic extension:



$$= \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos nx}_{\text{Fourier Cosine Series for } f \text{ on } [0,\pi]}$$

Next time: Bessel's Ineq.

$$f = \sum_{n=1}^{\infty} c_n \varphi_n$$

where $$\langle e_n, e_m \rangle = \int_{-\pi}^{\pi} e_n e_m \, dx$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Bessel's: $$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_{-\pi}^{\pi} |f|^2 \, dx$$

Consequently: Fourier coeff $\to 0$ as $n \to \infty$.

Parseval's Formula: Bessel's with $=$

(Holds for Fourier Series.)