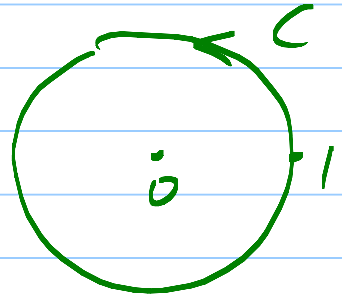
Lecture 40 Vibrating circular membrane, part 2

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$



$u = 0$ on C

$$\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{cases}$$

Try $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

$$\frac{T''}{T} = \boxed{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda}$$

$$\boxed{\text{T-prob: } T'' - \lambda T = 0}$$

Multiply red box by r^2:

$$\frac{r^2 R''}{R} + \frac{r R'}{R} - \lambda r^2 = \underbrace{-\frac{\Theta''}{\Theta} = \mu}$$

Periodic sol^{n}s:

$\Theta(-\pi) = \Theta(\pi)$

$\Theta'(-\pi) = \Theta'(\pi)$

Θ-prob: $\Theta'' + \mu\Theta = 0$

Periodic Boundary Conditions

Sol^n: $\mu = n^2$. Fourier Series!

$$\Theta_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$$

R-prob: $R(1) = 0$ ← B.C.

$$\frac{r^2 R''}{R} + \frac{rR'}{R} - \lambda r^2 = n^2 \leftarrow \mu$$

$$\boxed{\begin{gathered} r^2 R'' + rR' + (-\lambda r^2 - n^2)R = 0 \\ R(1) = 0 \end{gathered}}$$

Bessel's Eqn of order n after scaling the r-variable.

Get $R_m(r) = J_n(\beta_m r)$

where the β_m are zeroes of $J_n(s)$.

Sturm-Liouville : R_m's are $\perp$ on $[0,1]$ with repect to a weight fcn

Answer :

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} J_n(\beta_m r) \cos n\theta \cos \beta_m t$$
$$+ B_{nm} J_n(\beta_m r) \sin n\theta \cos \beta_m t$$
$$+ \cdots\cdots \sin \beta_m t$$
$$+ \cdots\cdots \sin \beta_m t$$

Use I.C. and orthogonality to get all coeff's.

Next problem : Heat Equation on a ball.

$$\frac{\partial u}{\partial t} = \Delta u \leftarrow \text{In spherical coordinates.}$$

$$u(\rho, \theta, \psi) = P(\rho)\, \Theta(\theta)\, \Psi(\psi)$$

$P(\rho)$ - probs : Legendre Eqn !

Orthogonal Sol^{n}s are the Legendre Polynomials!

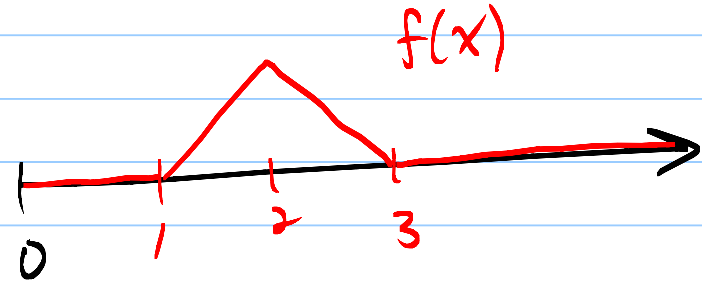
Hints:

1.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$x \geq 0$

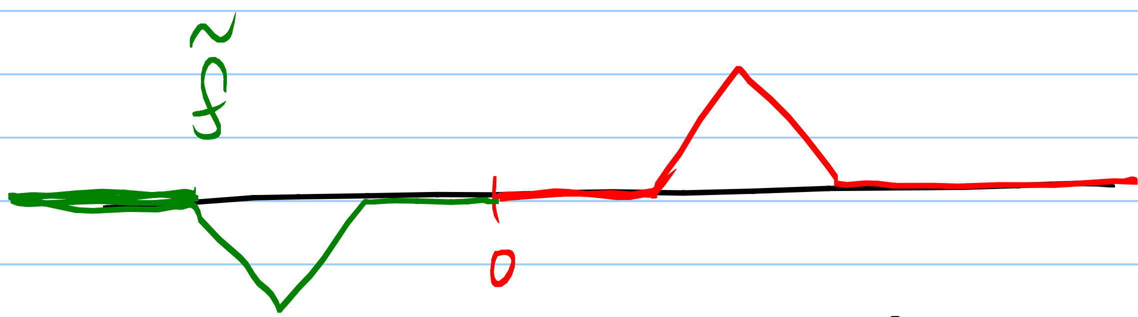
$t \geq 0$



$u(x,0) = f(x)$ I. C.

$u(0,t) = 0$ B. C.

Hint: Odd extension!



Big Idea: Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}$

with $u(x,0) = \tilde{f}(x)$

Check that $u(0,t) = 0$ for all t.

Also note that

$$\hat{f}(s) = \int_{-\infty}^{\infty} \underbrace{\underset{\substack{\uparrow \\ \text{odd}}}{f(x)}\, \underbrace{e^{-2\pi i s x}}_{\underset{\substack{\uparrow \\ \text{even}}}{\cos 2\pi s x} + i \underset{\substack{\uparrow \\ \text{odd}}}{\sin 2\pi s x}}}_{\int_{-R}^{R} = 0} \, dx$$

$$= 2i \int_0^{\infty} f(x) \sin 2\pi s x \, dx$$

p.95 : 2.d.

Fourier Series :

Féjér's Thm + Bessel's $\Rightarrow$ Parseval's

Aha! We have Weierstraß Thm about density of polys in the context of Legendre Poly's.

Best approx in L^2.

Practice Problems for next week

p. 91–92 : 14, 15, 16

Compact sets in $\mathbb{R}$: The closed and bounded sets.

closure

$$\operatorname{Supp} \varphi = \overline{\{x \in \mathbb{R} : \varphi(x) \neq 0\}}$$

Compact support means

$$\operatorname{Supp} \varphi \subset [a,b] \text{ for some } a, b.$$

OK to assume $[a,b] = [0, 1/2]$ in prob. 1.