Lecture 41 Fourier Series Review

Final Exam: Wed, May 3, 1-3 pm here.
Two sided handwritten crib sheet.

p. 91: 14. If f is C^1-smooth 2π-periodic fcn, then its F.S. converges absolutely to f.

1) Parseval's: If f is continuous 2π periodic, then

$$\sum |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

Best L^2-approximation property of F.S.

$$\left\| f - \sum_{-N}^{N} \hat{f}(n) e^{inx} \right\|^2$$

$$= \|f\|^2 - \sum_{-N}^{N} |\hat{f}(n)|^2 \leq \left\| f - \sum_{-N}^{N} b_n e^{inx} \right\|^2$$

2) $\widehat{f'}(n) = in\hat{f}(n)$

$$\widehat{f'}(n) = \int_{-\pi}^{\pi} f'(x)\, e^{-inx}\, dx$$

$= \ldots$ Integrate by parts

1+2 : $$\sum_{-\infty}^{\infty} |in\hat{f}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2\, dx$$

Want: $$\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$$

Then $$|\hat{f}(n)\underbrace{e^{inx}}_{|e^{inx}|=1}| \le |\hat{f}(n)| = M_n$$

Continuous fcn

$$\sum M_n < \infty$$

Weierstraß M-test yields that $\sum \hat{f}(n)e^{inx}$ converges <u>uniformly</u>

(to a <u>continuous</u> limit)

3) How to get $\sum |\hat{f}(n)| < \infty$ from $\sum |n\hat{f}(n)|^2 < \infty$.

Key: Cauchy-Schwarz:

$$|\langle v_1, v_2 \rangle| \le \|v_1\| \cdot \|v_2\|$$

$$\left| \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx \right| \le \sqrt{\int_{-\pi}^{\pi} |f|^2 dx} \sqrt{\int_{-\pi}^{\pi} |g|^2 dx}$$

$$\left| \sum_{n=1}^{N} a_n \overline{b_n} \right| \le \sqrt{\sum_1^N |a_n|^2} \cdot \sqrt{\sum_1^N |b_n|^2}$$

Hmmm. $\sum_{n=-N}^{N} n^2 |\hat{f}(n)|^2 < \infty \leftarrow$ know

$\sum |\hat{f}(n)| < \infty \leftarrow$ want

$$\sum_{\substack{n=-N \\ n\neq 0}}^{N} \underbrace{\frac{1}{n}}_{a_n} \cdot \underbrace{n|\hat{f}(n)|}_{b_n} \leq \underbrace{\sqrt{\sum_{-N}^{N} \frac{1}{n^2}}}_{<2\sqrt{\frac{\pi^2}{6}}} \cdot \underbrace{\sqrt{\sum_{-N}^{N} n^2 |\hat{f}(n)|^2}}_{<\infty \text{ by Parseval's for } f'}$$

Related fact: $L^1[a,b] \subset L^2[a,b]$.

$$\int_a^b |f(x)|\,dx = \int_a^b 1 \cdot |f(x)|\,dx$$

$$\leq \sqrt{\int_a^b 1^2\,dx} \cdot \sqrt{\int_a^b |f(x)|^2\,dx}$$

$$\|f\|_{L^1} \leq \sqrt{b-a}\,\|f\|_{L^2}$$

p. 15: f 2π periodic Lip_α $0<\alpha\leq 1$

$$|f(x)-f(y)| \leq C|x-y|^\alpha$$

Want $|\hat{f}(n)| \leq \dfrac{M}{n^\alpha}$ for some $M>0$.

a) $$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\underbrace{x+\tfrac{\pi}{n}}_{u})\, e^{-inx}\, dx$$

$u = x + \frac{\pi}{n}$ $\qquad$ $x = u - \frac{\pi}{n}$

$du = dx$

When $x = -\pi$: $\quad u = -\pi + \frac{\pi}{n}$

$x = \pi$: $\quad u = \pi + \frac{\pi}{n}$

$$I = -\frac{1}{2\pi}\int_{-\pi+\frac{\pi}{n}}^{\pi+\pi/n} f(u)\, \underbrace{e^{-in(u-\frac{\pi}{n})}}_{e^{-inu}\,\underbrace{e^{in\pi/n}}_{e^{i\pi}=-1}}\, du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\, e^{-inu}\, du = \hat{f}(n)$$

shift because integrand is 2π-periodic.

$\hat{f}(n)$ = Average of two ways :

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\tfrac{\pi}{n})\, e^{-inx}\, dx\right]$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[f(x) - f(x+\tfrac{\pi}{n})\right] e^{-inx}\, dx$$

So $$|\hat{f}(n)| \le \frac{1}{4\pi}\int_{-\pi}^{\pi} \underbrace{\left|f(x) - f(x+\tfrac{\pi}{n})\right|}_{\le c\left(\frac{\pi}{n}\right)^{\alpha}} \underbrace{\left|e^{-inx}\right|}_{=1} dx$$

Done! $M = \frac{c\pi^{\alpha}}{4\pi} \cdot 2\pi$

p. 16. $\alpha = 1$ case is even better.

We just got $|\hat{f}(n)| \le \frac{M}{n}$ if

f is Lip_1 $(\alpha = 1)$. But $\sum \frac{1}{n} = \infty$,

so looks like f is not absolutely

convergent. But it is!

Want to show $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$ if

f is Lip_1, meaning $|f(x)-f(y)| \le c|x-y|$.

a) $g_h(x) = f(x+h) - f(x-h)$

want →
$$\underbrace{\frac{1}{2\pi}\int_0^{2\pi} |g_h(x)|^2\,dx}_{\text{Parseval!}} = \underbrace{\sum_{n=-\infty}^{\infty} 4|\sin nh|^2 |\hat{f}(n)|^2}_{\sum |\hat{g}_h(n)|^2} \checkmark$$

$$\hat{g}_h(n) = e^{-inh}\hat{f}(n) - e^{inh}\hat{f}(n)$$

$$(2i)\left(-\left[\frac{e^{inh} - e^{-inh}}{2i}\right]\right)\hat{f}(n)$$

$$|\hat{g}_h(n)| = 2\,|\sin nh|\,|\hat{f}(n)|$$