Lecture 42

Total score	Pre Final E. grade
286 - 300	A+
276 - 285	A
270 - 275	A-
250 - 269	B+
219 - 249	B
200 - 218	B-
150 - 199	C
100 - 149	D
< 100	F

p. 16 on p. 92: f 2π periodic, Lip_1.

$$|f(x) - f(y)| \le M|x-y|$$

Then $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$ and so F.S. converges to f absolutely and uniformly by the W. M-Test.

a) Let $g_h(x) = f(x+h) - f(x-h)$

$$\hat{g}_h(n) = e^{inh}\hat{f}(n) - e^{-inh}\hat{f}(n)$$

Parseval's: $\frac{1}{2\pi}\int_0^{2\pi} |g_h|^2\, dx =$

$$\sum_{n=-\infty}^{\infty} \left| (2i)\cdot \frac{e^{inh}-e^{-inh}}{2i} \right|^2 |\hat{f}(n)|^2$$

$$4 = \sum_{n=-\infty}^{\infty} |\sin nh|^2 \, |\hat{f}(n)|^2$$

But $|g_h(x)| = \left| f(x+h) - f(x-h) \right|$

$$\leq M \underbrace{\left| (x+h) - (x-h) \right|}_{2hM}$$

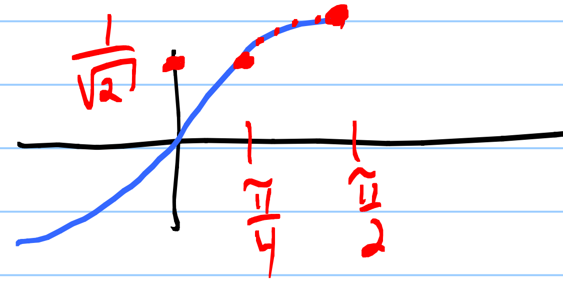
Get $\boxed{\displaystyle\sum_{n=-\infty}^{\infty} |\sin nh|^2 \, |\hat{f}(n)|^2 \leq K^2 h^2}$

b) Let p be positive integer. Let $h = \frac{\pi}{2^{p+1}}$. Show that

$$\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2 \pi^2}{2^{2p+1}}$$

$$|n| = 2^{p-1} + k \quad , \quad k = 1, 2, \dots, 2^{p-1}$$

$$\sin \underbrace{(2^{p-1}+k)\frac{\pi}{2^{p-1}}}_{\frac{\pi}{4} + k\frac{\pi}{2^{p-1}}}$$



$$= \frac{\pi}{4} + \frac{\pi}{4} \text{ when } k = 2^{p-1}$$

Aha! $|\sin^2| \ge \left(\frac{1}{\sqrt{2}}\right)^2$

$$\frac{1}{2}\sum_{\text{Range}} |\hat{f}(n)|^2 \le \sum_{\text{Range}} |\sin nh|^2 \, |\hat{f}(n)|^2$$

$$\le \sum_{n=-\infty}^{\infty} |\sin nh|^2 \, |\hat{f}(n)|^2$$

$$\le K^2 \Big(\underbrace{\frac{\pi}{2^{p+1}}}_{h}\Big)^2$$

So $\displaystyle\sum_{\text{Range}} |\hat{f}(n)|^2 \le \frac{K^2 \pi^2}{2^{2p+1}}$

c) Use Cauchy-Schwarz :

$$\sum_{\text{Range}} |\hat{f}(n)| = (1,1,\dots,1)\cdot\left(|\hat{f}(n)|_{n=\text{Range}}\right)$$
$$\le \left(\sum 1^2\right)^{1/2}\left(\sum|\hat{f}(n)|^2\right)^{1/2}$$
$$\le \left(2^{p-1}\right)^{1/2}\left(\frac{K^2\pi^2}{2^{2p+1}}\right)^{1/2}$$

Finally, add up all $\sum_{\text{Range}}$ over p and compare to a geom. series!

d) Get same result for Lip_α fcns $\frac{1}{2} < \alpha \le 1$. [Bernstein's Thm.]

Heisenberg uncertainty principle:

Prob. 2 on F. Exam, part 1.

f and $\hat{f}$ cannot both have compact support.

[Support of f = Closure of the

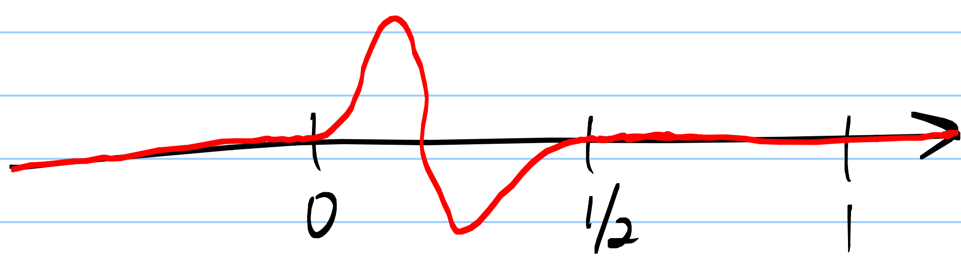
set where $f(x) \neq 0$.]

Step 1: Suppose f and $\hat{f}$ have compact support.

Facts: $$\widehat{f(x+h)} = e^{2\pi i h s}\hat{f}(s)$$

$$\widehat{f(cx)} = \frac{1}{c}\hat{f}\left(\frac{s}{c}\right)$$

So we may translate and scale to assume f is supported in $[0, 1/2]$.



Suppose $\hat{f}$ is zero for $|s| > N$.

Consider Fourier Series for f on $[0,1]$:

$$a_n = \text{"}\hat{f}(n)\text{"} = \int_0^1 f(x)\, e^{-i2\pi n x}\, dx$$

Aha! $$= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi n x}\, dx$$

$$= \hat{f}(n) = 0, \ |n| > N.$$

F. Transform!

Conclude that $f(x) = \sum_{n=-N}^{N} a_n e^{i2\pi n x}$ is

a trig poly on $[0,1]$. But f

vanishes on $(\frac{1}{2}, 1)$.

Fact: A trig poly that vanishes on an open set must $\equiv 0$.

Why: $e^{i2\pi n x} = \underbrace{\cos 2\pi n x}_{} + i \underbrace{\sin 2\pi n x}_{}$

= Convergent power series with R. of C. = ∞.

Taylor's Formula: $A_n = \frac{f^{(n)}(3/4)}{n!}$ at

center $x_0 = \frac{3}{4}$ $\equiv 0$ for all n.

Get the zero power series!

So $f \equiv 0$. $\hat{f} \equiv 0$. $\lightning$.

Or: $$\hat{f}(s) = \int_{-\infty}^{\infty} \underbrace{f(x)} e^{-i2\pi s x}\, dx$$

$$= \int_0^1 \sum_{-N}^{N} a_n e^{i2\pi n x} e^{-i2\pi s x}\, dx$$

$$= \sum_{-N}^{N} \frac{1}{2\pi i(n-s)} a_n$$

boom!
at $n=s$.
So $a_n = 0$.