# Lecture 5 Bessel's Ineq

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Fun: $e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

!‖ $e^{i\alpha}e^{i\beta} =$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) =$$

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

Fact: $e^{i\alpha+i\beta} = e^{i\alpha}e^{i\beta}$

$$e^{in\theta}\cdot e^{-in\theta} = e^0 = 1$$

$$\int_a^b e^{in\theta}\,d\theta = \int_a^b \cos n\theta\,d\theta + i\int_a^b \sin n\theta\,d\theta$$

$$= \left[\frac{1}{n}\sin\theta - \frac{1}{n}i\cos n\theta\right]\Big|_a^b$$

$$= \frac{1}{in}e^{in\theta}\Big|_a^b$$

Same as $e^{kx}$, $\mathbb{R}$.

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

<u>Why:</u> $c_n = \frac{1}{2}(a_n - i\, b_n)$

$$= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx - i\, \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \underbrace{\left[\cos nx - i \sin nx\right]}_{e^{-inx}} dx$$

$$e^{-inx} = \overline{e^{inx}}$$

$$\overline{(x+iy)} = x - iy$$

<u>Bessel's Ineq:</u> On $[-\pi, \pi]$ suppose $\varphi_n$ are continuous fcns on $[-\pi, \pi]$ with

$$\langle \varphi_n, \varphi_m \rangle = \int_{-\pi}^{\pi} \varphi_n(x)\, \overline{\varphi_m(x)}\, dx$$

$$= \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

orthonormal

$$\|f\|^2 = \langle f, f \rangle = \int_{-\pi}^{\pi} |f|^2 \, dx$$

Big question: If $\quad f = \sum_{n=1}^{\infty} c_n \varphi_n,$

then

$$\int_{-\pi}^{\pi} f(x)\varphi_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_{-\pi}^{\pi} \varphi_n \varphi_m \, dx$$

$$= c_m \underbrace{\int_{-\pi}^{\pi} \varphi_m^2 \, dx}_{1}$$

So $\boxed{c_m = \langle f, \varphi_m \rangle}$

If we take $c_m = \langle f, \varphi_m \rangle$, do we get $\quad f = \sum c_m \varphi_m$ ?

Hmmm. Yes for $f = \sum_{n=1}^{\infty} c_n \sin nx$ on $[0, \pi]$. But if I tossed out a few $\sin mx$'s, then I could no longer expand every $f$!

<u>Def$^n$</u>: $\{\varphi_n\}_{n=1}^{\infty}$ is called <u>complete</u> if every fcn can be expanded.

<u>Facts</u>: a) $\{\sin nx\}_{n=1}^{\infty}$ on $[0, \pi]$ are complete.

b) $1$, $\cos nx$, $\sin nx$ are complete on $[-\pi, \pi]$.

<u>Bessel's</u>: Let $c_m = \langle f, \varphi_m \rangle$.

$$\overset{E}{E^*} = \left\| f - \sum_{n=1}^{N} \overset{A_n}{c_n} \varphi_n \right\|^2 =$$

$\geq 0$

$$\int_{-\pi}^{\pi}\Big(f(x)-\sum_{n=1}^{N}\overset{A_n}{c_n}\varphi_n(x)\Big)^2\,dx =$$

$$\int_{-\pi}^{\pi}\Big(f(x)-\sum_{n=1}^{N}\overset{A_n}{c_n}\varphi_n(x)\Big)\Big(f(x)-\sum_{m=1}^{N}\overset{A_m}{c_m}\varphi_m(x)\Big)\,dx$$

$$= \int_{-\pi}^{\pi} f^2\,dx - \sum_{n=1}^{N}\overset{A_n}{c_n}\underbrace{\int_{-\pi}^{\pi}\varphi_n f\,dx}_{c_n} - \sum_{m=1}^{N}\overset{A_m}{c_m}\underbrace{\int_{-\pi}^{\pi}\varphi_m f\,dx}_{c_m}$$

$$+\sum_{n=1}^{N}\sum_{m=1}^{N}\overset{A_n}{c_n}\overset{A_m}{c_m}\underbrace{\int_{-\pi}^{\pi}\varphi_n\varphi_m\,dx}_{\delta_{nm}}$$

$$= \|f\|^2 - \sum_{1}^{N} c_n^2 - \sum_{1}^{N} c_m^2 + \sum_{1}^{N} c_n^2$$

Aha! $\quad 0 \le \|f - \sum c_n\varphi_n\|^2 = \|f\|^2 - \sum_{n=1}^{N} c_n^2$

So $\boxed{\sum_{n=1}^{N} c_n^2 \le \|f\|^2}$ Bessel's!

Let $N \to \infty$.

Consequence: Fourier coeff $\to 0$ as $n \to \infty$.

$$E - E^* = \sum_{n=1}^{N} (A_n - c_n)^2$$

sum of squares $\geq 0$

Theorem: The (generalized) fourier coeff $c_n = \langle f, \varphi_n \rangle$ make the $L^2$ error $\| f - \sum_{n=1}^{N} c_n \varphi_n \|^2$ as small as possible. Changing any $c_n$'s to $A_n$'s makes the error strictly bigger.

Fact: N-th partial sum of a Fourier Series is the best approx to f among all N-th degree trig polys.

Fact: Bessel's Ineq becomes an equality exactly when $\{e_n\}_1^\infty$ form a complete orthonormal basis for $L^2[-\pi,\pi]$. (Parseval's Identity.)

Hilbert space!

$$\vec{v} = c_1\vec{u}_1 + \cdots + c_N\vec{u}_N$$

Finite dim vector space $\quad \langle \vec{u}_n, \vec{u}_m \rangle = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

Orthog basis: $\{u_n\}_1^N$.

$$\vec{v}\cdot\vec{u}_m = \sum_1^N c_n \vec{u}_n \cdot \vec{u}_m = c_m \cdot 1$$

$$\boxed{c_m = \vec{v}\cdot\vec{u}_m} \leftarrow c_m \text{ coordinate}$$

$$L^2[-\pi,\pi] = \left\{ f : \int_{-\pi}^{\pi} f^2\,dx < 0 \right\}$$

is an infinite dim$^{l}$ inner product

and $e^{inx}$ $n = 0, \pm 1, \pm 2, \ldots$

are a complete orthogal basis.

Think: Fourier Coeff = Coords