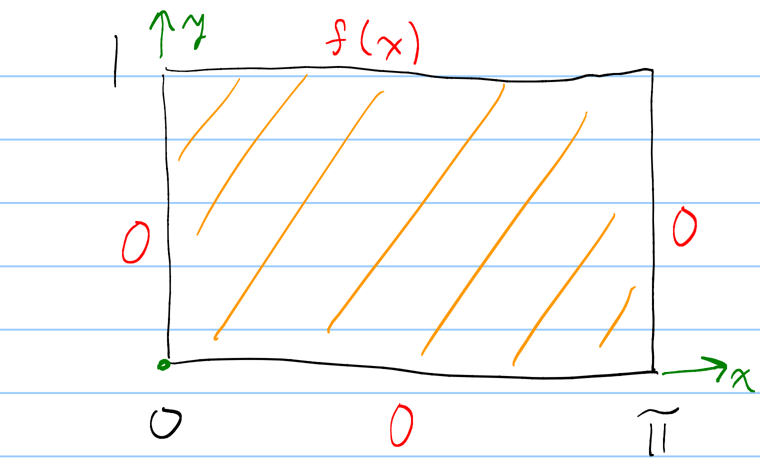
Lecture 6

Dirichlet prob



$$\Delta u = 0$$

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \right)$$

Step 1: Try separation of variables:

$$u(x,y) = X(x)\,Y(y)$$

want

$$\frac{\partial^2}{\partial x^2}\left[X(x)\,Y(y)\right] + \frac{\partial^2}{\partial y^2}\left[X(x)\,Y(y)\right] = 0$$

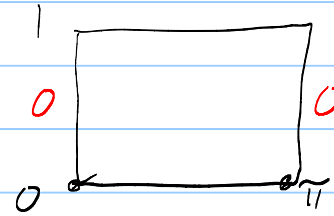
$$X''Y + XY'' = 0$$

$$X''Y = -XY''$$

$$\boxed{\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \quad \text{a const!}}$$

Step 2: Boundary conditions

$$u(x,y) = X(x)\,Y(y)$$



Left side $\quad u(0,y) = \underset{\text{need}=0}{\underbrace{X(0)}}\, \underbrace{Y(y)}_{\text{don't want just zero sol}^n!} \overset{\text{want}}{=} 0$

Right side $\quad u(\pi,y) = \underset{\text{need}=0}{\underbrace{X(\pi)}}\, Y(y) \overset{\text{want}}{=} 0$

<u>X-prob</u>: $X'' - \lambda X = 0$

$$X(0) = 0,\quad X(\pi) = 0$$

Sturm-Liouville prob! Same as harpsichord prob.

<u>$\lambda = 0$</u>: Only got zero sol^n

<u>$\lambda > 0$</u>: Only got zero sol^n

<u>$\lambda < 0$</u>: $\lambda = -n^2 \qquad X_n(x) = \sin nx$

<u>Step 3</u>: Now solve $Y(y)$ prob. <u>$\lambda = -n^2$</u>

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = -n^2$$

2nd order linear const coeff ODE :

$$ay'' + by' + cy = 0$$

Try $y = e^{rx}$.

$$a \cdot r^2 e^{rx} + b \cdot r e^{rx} + c e^{rx} = 0$$

want

$$\underbrace{(ar^2 + br + c)}_{\text{need } = 0} e^{rx} = 0$$

never zero

Get two roots: r_1, r_2

Case: r_1, r_2 real, $\neq$. $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$(b^2 - 4ac > 0)$

Aha! $e^{r_1 x}$, $e^{r_2 x}$ solve ODE.

So is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Are these all? Can we solve any and all intial conditions?

Initial conditions: $y(0) = A$
$y'(0) = B$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$y' = c_1 r_1 e^{r_1 x} + c_2 r_2 e^{r_2 x}$$

want

$$\begin{cases} y(0) = c_1 + c_2 = A \\ y'(0) = c_1 r_1 + c_2 r_2 = B \end{cases}$$

$$\begin{cases} c_1 + c_2 = A \\ c_1 r_1 + c_2 r_2 = B \end{cases}$$

need $\det \neq 0$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix}}_{\det = r_2 - r_1 \neq 0 \text{ ! Yes!}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Can solve all I.C.'s.

E & U Thm: There exists a sol^n to ODE solving $\begin{cases} y(0) = A \\ y'(0) = B \end{cases}$ and it is unique.

We found gen^l sol^n: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case $r_1 = r_2$ (real) $\quad (b^2 - 4ac = 0)$

Get 1 sol^n: $y_1 = e^{r_1 x}$

Try $y_2 = u y_1$. Get first order ODE for u.

In end, find $y_2 = x e^{r_1 x}$.

Wronskian: $\det \begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} \neq 0$.

Gen^{l} Sol^{n} $\quad c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

Case $r = \alpha + i\beta$ complex $\quad (b^2 - 4ac < 0)$

Complex sol^{n}: $\quad e^{(\alpha + i\beta)x}$

$$= e^{\alpha x} e^{i\beta x}$$

$$= \underbrace{e^{\alpha x} \cos\beta x}_{} + i \underbrace{e^{\alpha x} \sin\beta x}_{}$$

two real sol^{n}s!

Wronskian $\neq 0$!

Get Gen^{l} Sol^{n} $\quad y = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$

$$Y'' - n^2 Y = 0$$

$$r^2 - n^2 = 0 \qquad \boxed{r = \pm n}$$

$$Y_n(y) = c_1 e^{ny} + c_2 e^{-ny}$$

Bottom side

Need $u(x,0) = X(x)\underbrace{Y(0)}_{\text{need } = 0} = 0$

want

$$Y_n(0) = c_1 + c_2 = 0 \qquad c_2 = -c_1$$

$$Y_n(y) = c_1 e^{ny} - c_1 e^{-ny} = \underbrace{c_1 \cdot 2}_{c} \cdot \frac{e^{ny} - e^{-ny}}{2}$$

$$Y_n(y) = c \sinh ny$$

Got for $n = 1, 2, 3, \ldots$

$$u_n(x,y) = c_n \sin nx \sinh ny$$

Big idea: Try $u(x,y) = \sum_{n=1}^{\infty} c_n \sin nx \sinh ny$

Last item: $u(x,1) = f(x)$

Answer: Chapter 1 : Problem 1