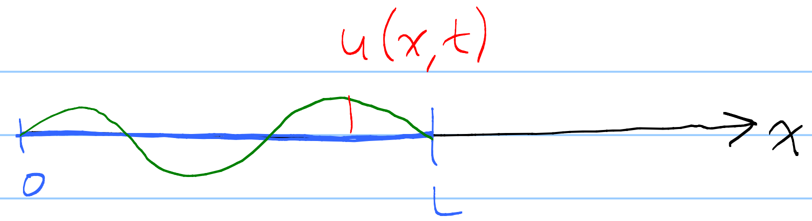
Lecture 7 Piano problem



Wave eqn: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

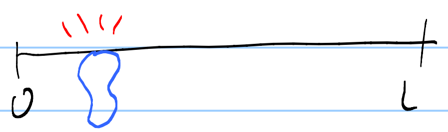
B.C.: $u(0,t) = 0$ all t

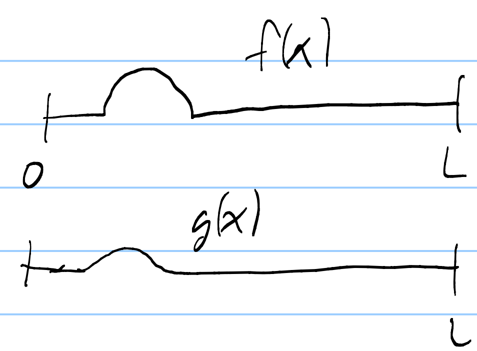
$u(L,t) = 0$ all t

Initial C.: $u(x,0) = f(x) \leftarrow$ given shape

$\dfrac{\partial u}{\partial t}(x,0) = g(x) \leftarrow$ given velocity profile

piano





Separation of vars: $u(x,t) = X(x)T(t)$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda$$

want

BC: $u(0,t) = \underbrace{X(0)}_{=0} T(t) = 0$

$u(L,t) = \underbrace{X(L)}_{=0} T(t) = 0$

Sturm-Liouville:

$$X''(x) - \lambda X(x) = 0$$

$$\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases}$$

Case $\lambda = 0$: No non-zero sol^{n}s

Case $\lambda > 0$: No non-zero sol^{n}s

Case $\lambda < 0$: $\lambda = -k^2$

$$X(x) = \cancel{c_1 \cos kx} + c_2 \sin kx$$

want

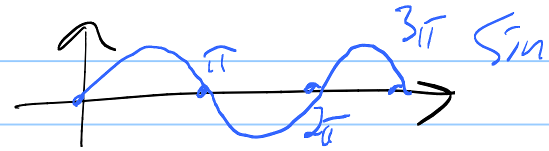
$$X(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1 = 0$$

want

$$X(L) = c_2 \sin kL = 0$$

don't want $c_2 = 0$ too! Zero sol^n!

Need $\boxed{\sin kL = 0}$



Need $kL = n\pi \quad n = 1, 2, 3, \ldots$

$$k = \frac{n\pi}{L}$$

For each n, get $X_n(x) = c \sin\frac{n\pi x}{L}$

Take $c = 1$.

Back to T prob:

$$\frac{1}{c^2}\cdot\frac{T''}{T} = \lambda = -\left(\frac{n\pi}{L}\right)^2$$

$$T'' + \left(\frac{cn\pi}{L}\right)^2 T = 0$$

Get $T_n(t) = A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi}{L}t$

Now $u_n(x,t) = X_n(x)\,T_n(t)$ and

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right)$$

Finally, IC:

want

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

Orthogonal on $[0,L]$!

Find $\boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}\right)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(-\frac{cn\pi}{L} A_n \sin\frac{cn\pi t}{L} + \frac{cn\pi}{L} B_n \cos\frac{cn\pi}{L} t\right)$$

want

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \underbrace{\left(\frac{cn\pi}{L} B_n\right)}_{\text{Fourier Sine Series coeff for } g} \sin\frac{n\pi x}{L} = g(x)$$

$$\boxed{\frac{cn\pi}{L} B_n = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx}$$

Trig identities:

$*$ $\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

Take $g(x) \equiv 0$. $f(x)$ = initial shape. B_n's $= 0$.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{cn\pi t}{L}$$

$$= \sum_{n=1}^{\infty} A_n \frac{1}{2}\left[\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\frac{n\pi}{L}(x-ct)\right]$$

Whoa! $f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]}$$

Jacob's Solution!

$$\frac{\partial}{\partial x} f(x+ct) = f'(x+ct)\cdot 1$$

$$\frac{\partial^2}{\partial x^2} f(x+ct) = f''(x+ct)$$

$$\frac{\partial}{\partial t} f(x+ct) = f'(x+ct)\cdot c$$

$$\frac{\partial^2}{\partial t^2} f(x+ct) = f''(x+ct)\cdot c^2$$

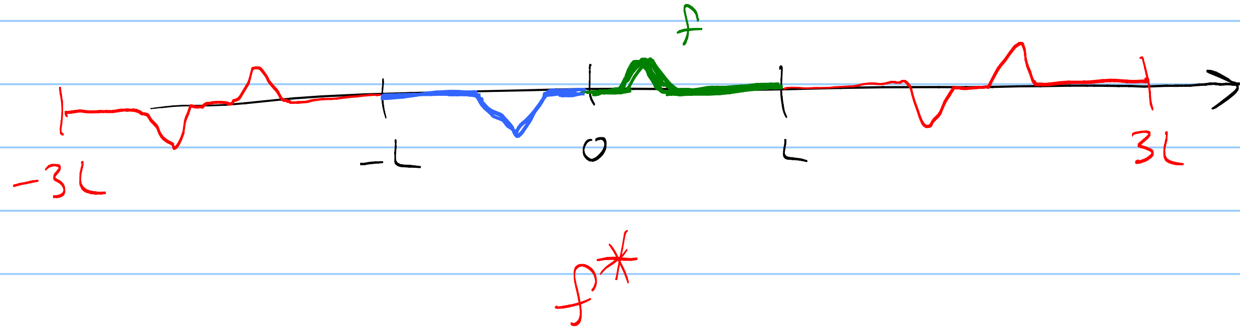
$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ✓

IC: $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$

$u(x,0) = \frac{1}{2}\left[f(x) + f(x)\right] = f(x)$ ✓

Hmmm. What happens when $x+ct$ or $x-ct$ leaves $[0,L]$.

Let f^* be the odd $2L$ periodic extension of f:



$$u(x,t) = \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct) \right]$$

How to get this in a straightforward way?

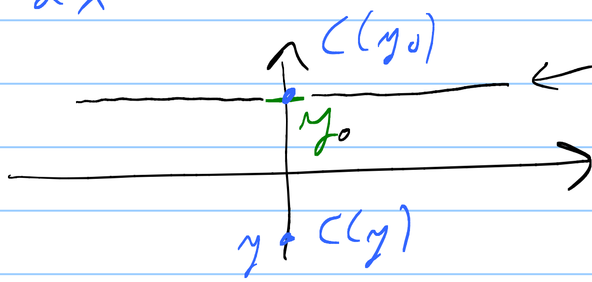
<u>D'Alembert's Method:</u> Change of vars:

$$u = x - ct \quad , \quad v = x + ct$$

Wave Eqn becomes $\frac{\partial^2}{\partial u \partial v} \overline{W} = 0$

<u>Step 1</u>: Solve easiest PDE:

$$\frac{\partial u}{\partial x} \equiv 0 \qquad u(x,y)$$



$\leftarrow \frac{\partial u}{\partial x}(x, y_0) \equiv 0$

See u = const on horizontal lines

Sol^{n}: $u(x,y) = C(y)$